A Bayesian estimator for the multifractal analysis of multivariate images

Herwig Wendt[1]

Collaborations: P. Abry[3], Y. Altmann[2], **S. Combrexelle**[1], N. Dobigeon[1], S. McLaughlin[2], J.-Y. Tourneret[1]

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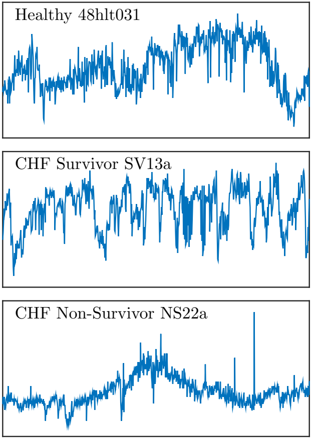
JFRB, Toulouse, 31 May 2018

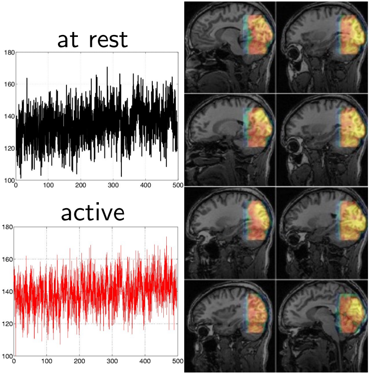
Empirical data: signals

Human Heart Rate



Collab. Y. Yamamoto (U Tokyo)
and K. Kiyono (U. Osaka)

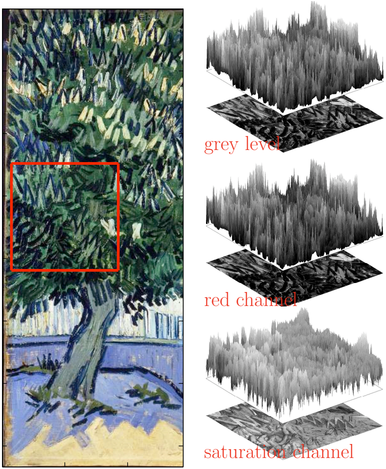
fMRI signals



Ph. Ciuciu (CEA)

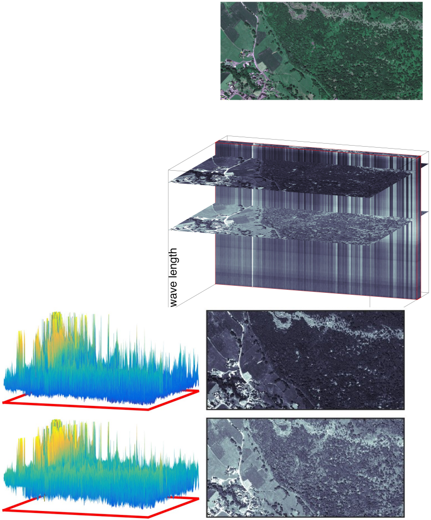
Empirical data: images

<u>Van Gogh's Painting</u>



Collab. Van Gogh Museum, Amsterdam

<u>Hyperspectral image</u>



[Sheeren'11]

Empirical data: images

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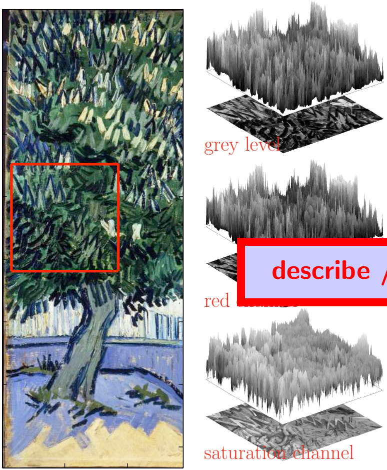

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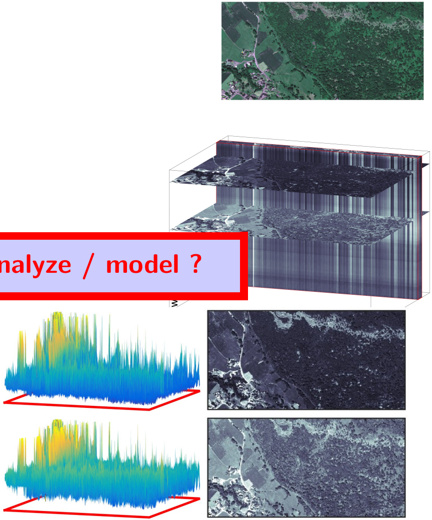


describe / analyze / model ?

[Sheeren'11]

Part 1: Multifractal analysis

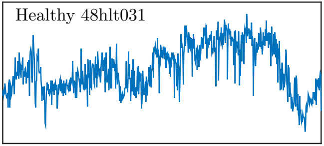
functional analysis, geometric description and higher-order statistics

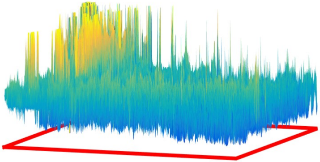
Part 2: Bayesian model for single image

from Gaussian random field to data augmented model

Part 3: Bayesian model for multivariate data

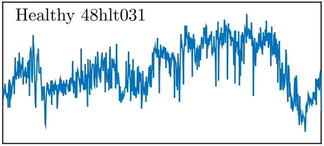
Markov field joint prior for multifractal parameters

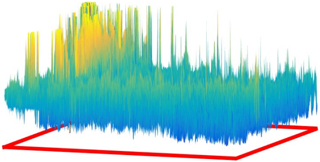




Part 1: Multifractal analysis

functional analysis, geometric description and higher-order statistics





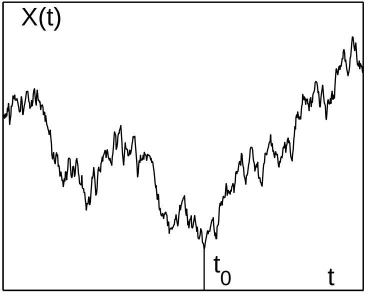
Multifractal spectrum

- Local regularity of $X(t)$ at t_0

  Hölder exponent

  $h(t_0) \triangleq \sup_\alpha \{\alpha : \; |X(t) - X(t_0)| < C|t - t_0|^\alpha\}$ $\qquad 0 < \alpha$

  $h(t_0) \to 1 \Rightarrow$ smooth, very regular,
  $h(t_0) \to 0 \Rightarrow$ rough, very irregular



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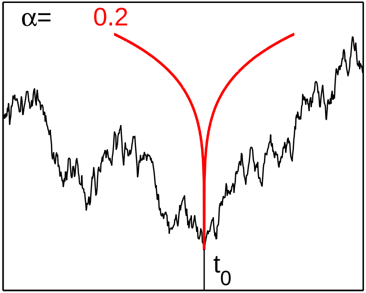

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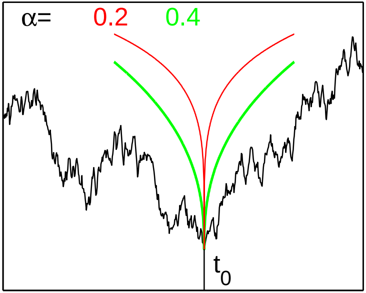

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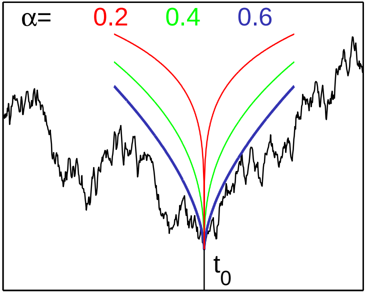

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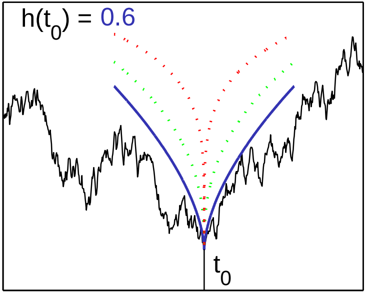

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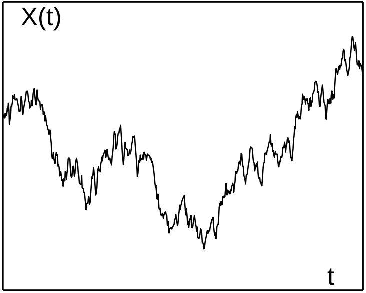
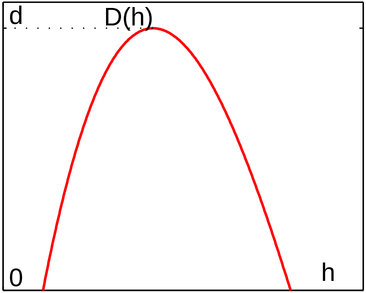
- Multifractal Spectrum $\mathcal{D}(h)$: Fluctuations of regularity $h(t)$
  - Set of points that share same regularity $\{t_i | h(t_i) = h\}$
  - Fractal (or Haussdorf) Dimension of each set:

$$\mathcal{D}(h) \triangleq \dim_H\{t : \ h(t) = h\}$$





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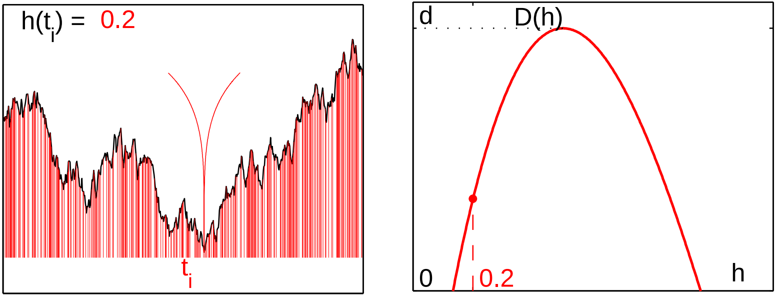

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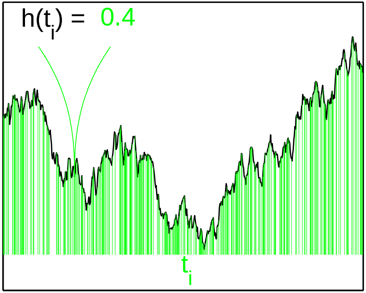

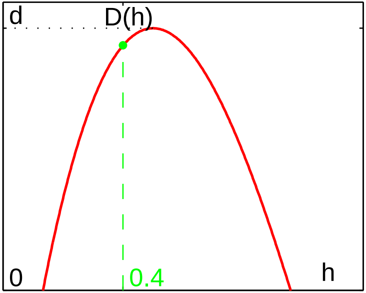

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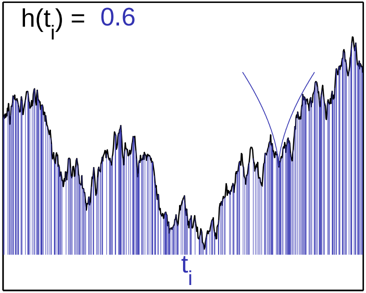

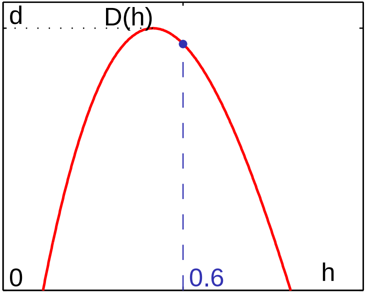

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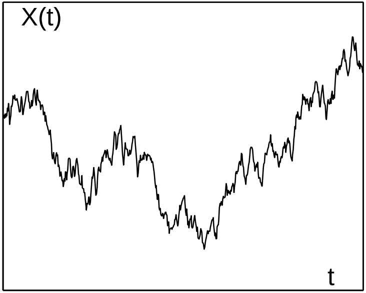
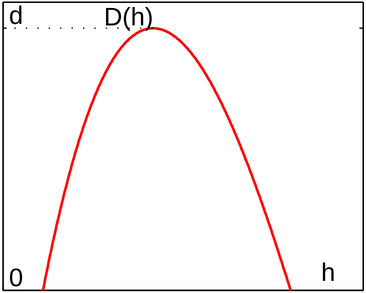
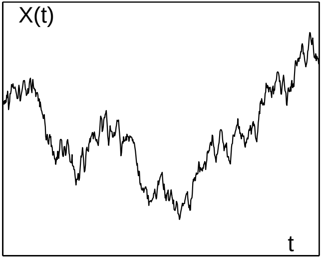


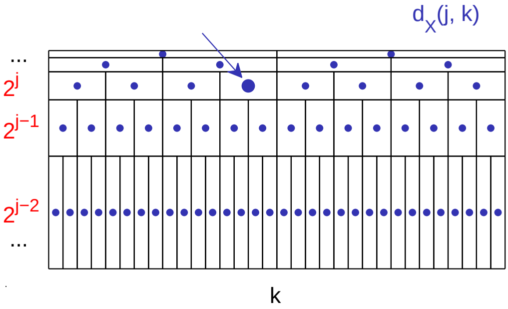


Estimation: Multifractal formalism

- $D(h)$ in practice $\rightarrow$ multifractal formalism [Parisi85]
- Multiresolution quantities

$d(j,k)$: DWT coefficient

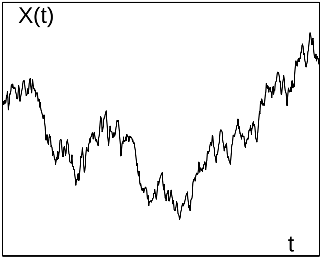


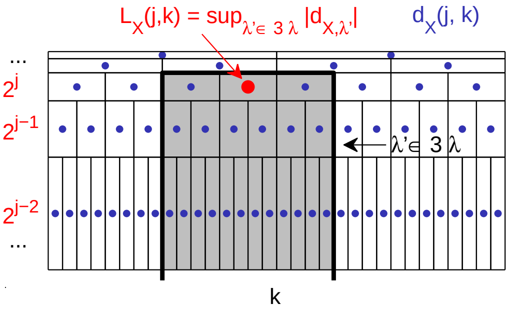


Estimation: Multifractal formalism

- $D(h)$ in practice $\rightarrow$ multifractal formalism [Parisi85]
- Multiresolution quantities: wavelet leaders $\{\ell(j\cdot,\cdot)\}$ [Jaffard04]

$$\ell(j,k) \triangleq \sup_{\lambda' \subset 3\lambda_{j,k}} |d(\lambda')|, \qquad d(j,k):\ \text{DWT coefficient}$$





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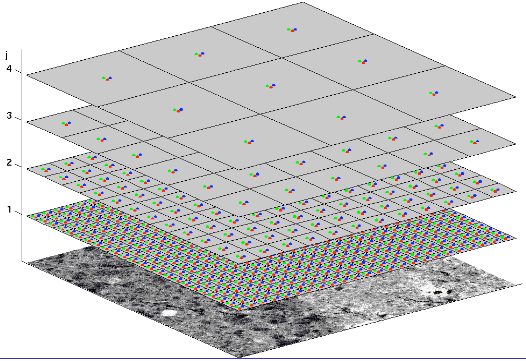

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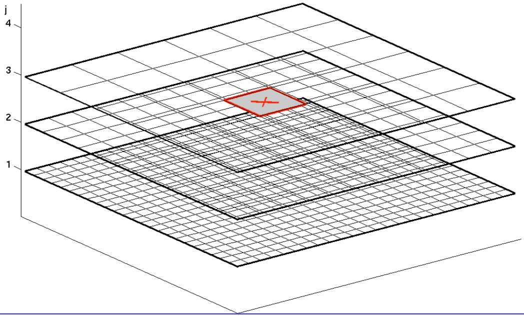

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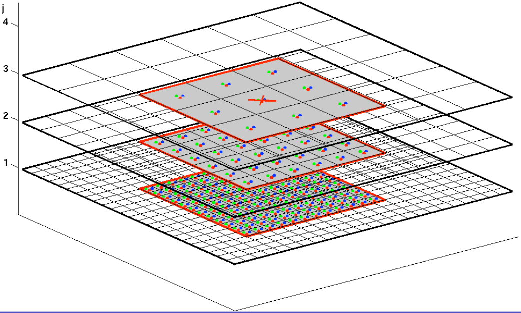

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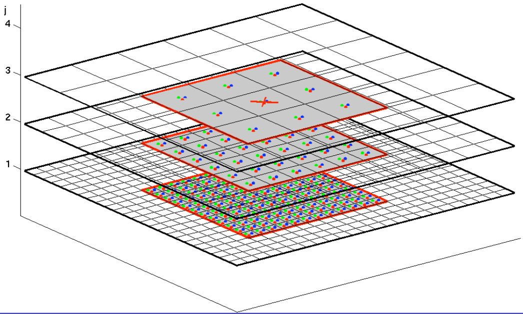
Key property: (under uniform regularity conditions)

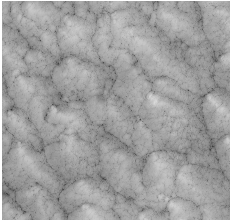
$$h(x_0) = \liminf_{j \to -\infty} \frac{\log\left(\ell_X(j,k(x_0))\right)}{\log(2^j)}$$



Estimation: Multifractal formalism

[Frisch,Parisi'85],[Jaffard'04]

$X \rightarrow d(a, k) \rightarrow \ell(a, k)$



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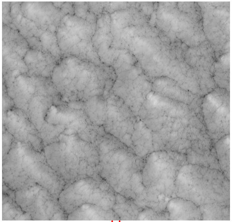

$\Downarrow$

$$S(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} \ell(a,k)^q$$

structure functions

Estimation: Multifractal formalism

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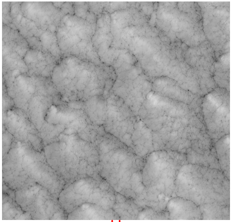

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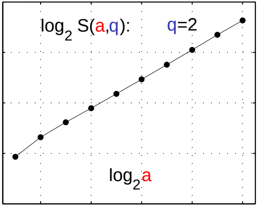

$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \rightarrow 0$

power laws

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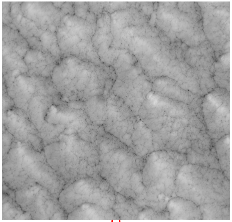

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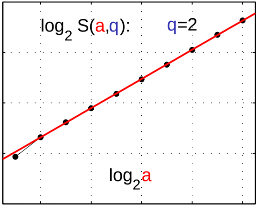

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power laws

$$\zeta(q) = \liminf_{a\to 0} \frac{\ln S(a,q)}{\ln a}$$

scaling function

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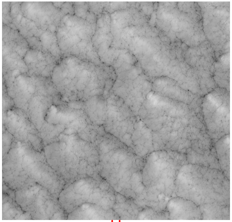

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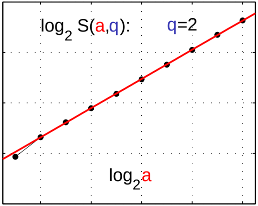
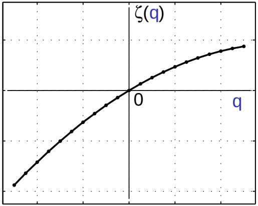


$\Longrightarrow$

$$\begin{aligned} S_n(a,q) &\simeq a^d \sum_h a^{-D(h)} a^{hq} \\ &\simeq \sum_h a^{d-D(h)+hq} \\ &\underset{a\to 0}{\sim} c_q a^{\zeta(q)} \end{aligned}$$

Saddle-point argument:
$\Rightarrow$ Legendre transform
$\zeta(q) = \min_{q\neq 0} (d + hq - D(h))$

$S(a,q) \simeq c_q a^{\zeta(q)}, \quad a \to 0$



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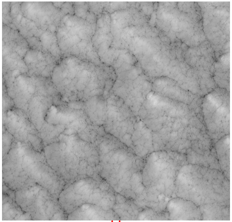

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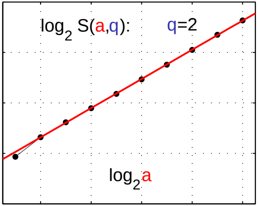

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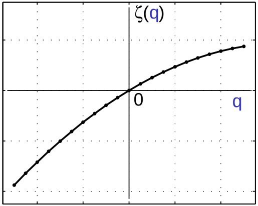

$$\Uparrow$$

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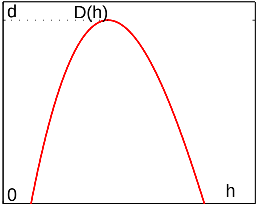
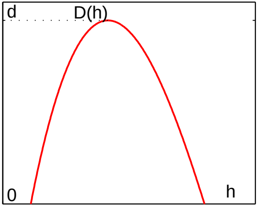


Multifractal spectrum and log-cumulants

- Polynomial expansion:

$$\zeta(q) = \sum_{p=1}^{\infty} c_p \frac{q^p}{p!}$$

$$D(h) = \min_{q \neq 0} \left(d + qh - \zeta(q) \right)$$



detailed

Multifractal spectrum and log-cumulants

- Polynomial expansion:

$$\zeta(q) = \sum_{p=1}^{\infty} c_p \frac{q^p}{p!}$$

$$\Rightarrow D(h) \approx d + \frac{c_2}{2!}\left(\frac{h - c_1}{c_2}\right)^2 - \frac{c_3}{3!}\left(\frac{h - c_1}{c_2}\right)^3 + \dots$$

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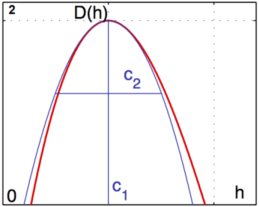

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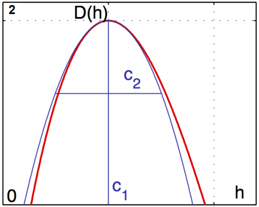

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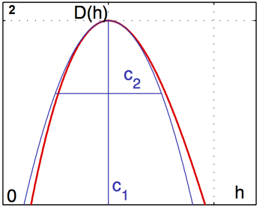

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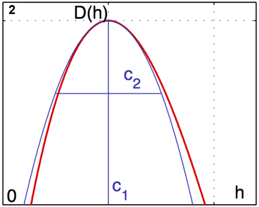

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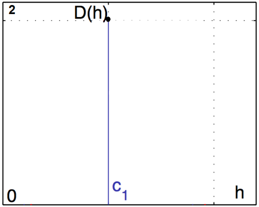

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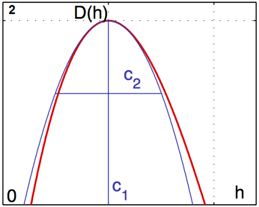

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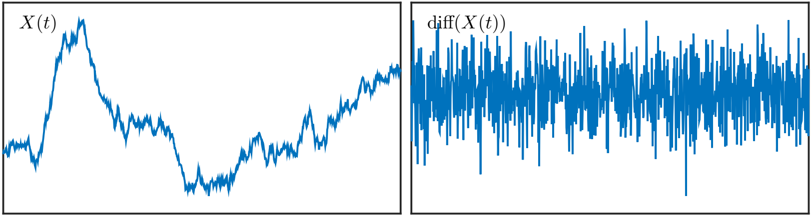
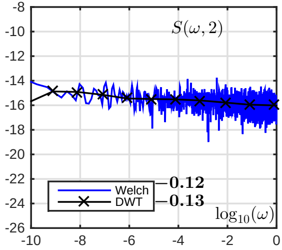
multifractal cascades $\rightarrow c_2 < 0$

Intuitions: self-similar properties

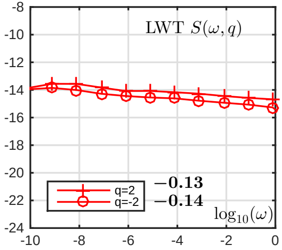
Changing the global regularity $\sim c_1$



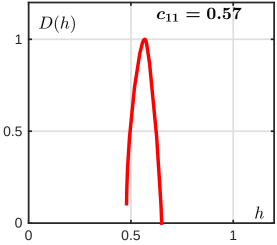
Spectrum, DWT ($q = 2$)



Leaders ($q = \pm 2$)



MF formalism



$\longrightarrow$ 2nd order properties

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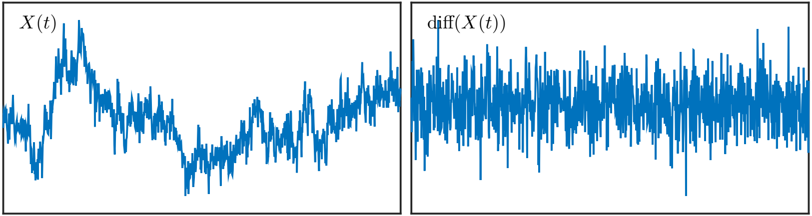
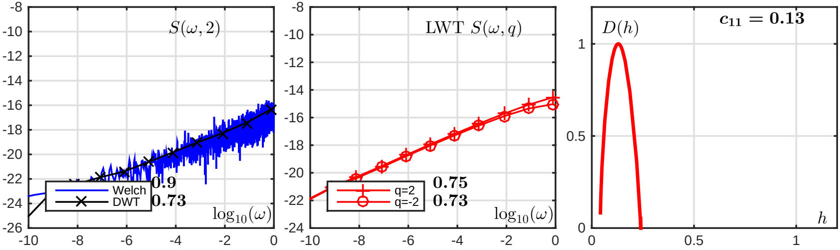


Spectrum, DWT ($q = 2$) | Leaders ($q = \pm 2$) | MF formalism



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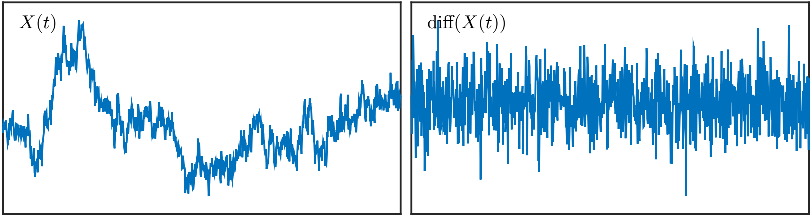
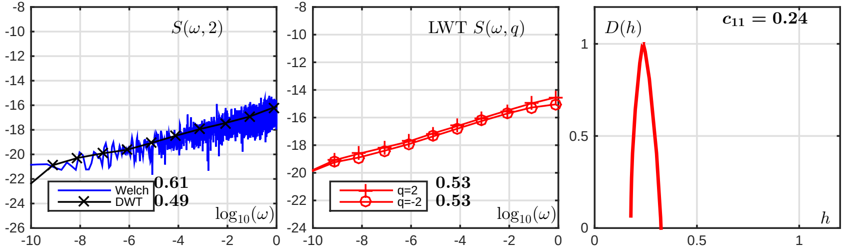


Spectrum, DWT ($q = 2$) | Leaders ($q = \pm 2$) | MF formalism



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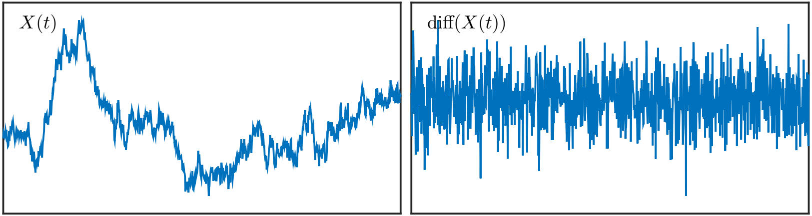
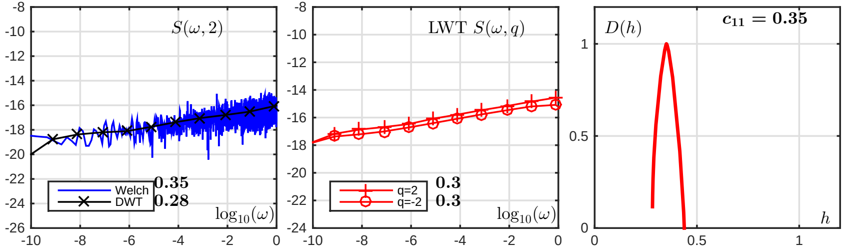


Spectrum, DWT ($q = 2$) Leaders ($q = \pm 2$) MF formalism



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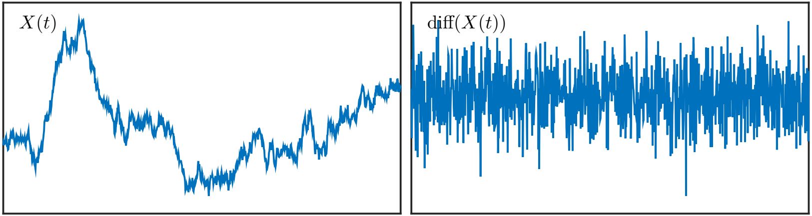

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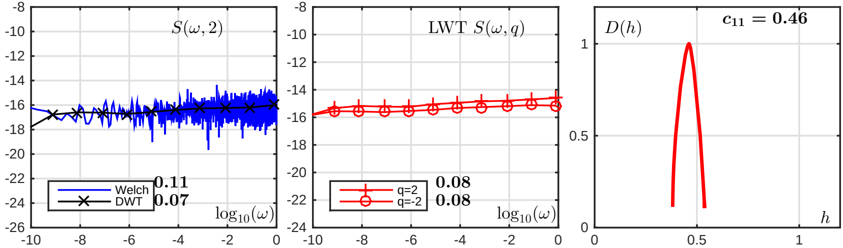

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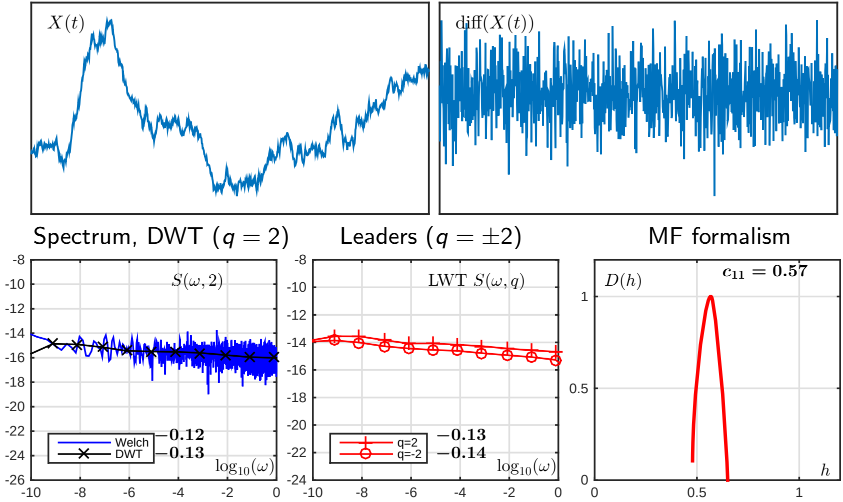

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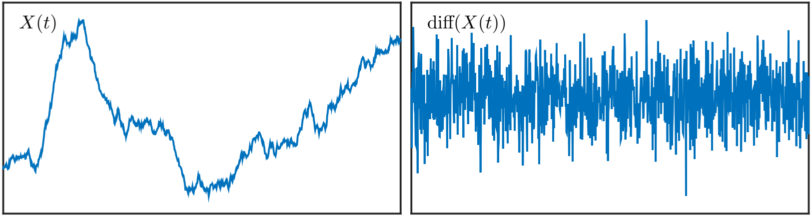

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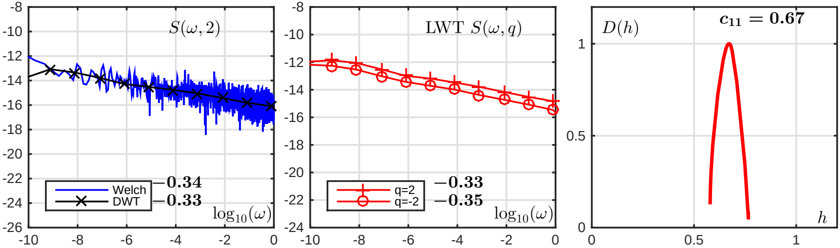

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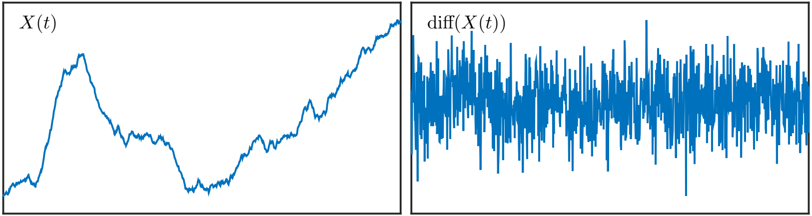

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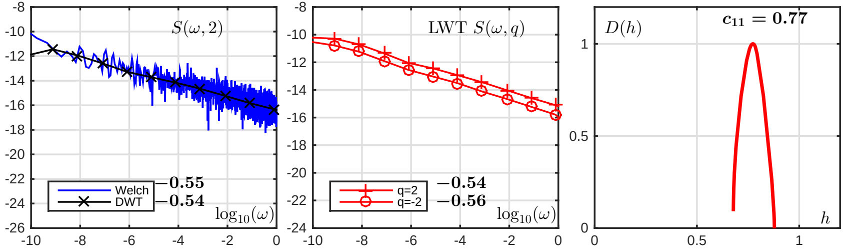

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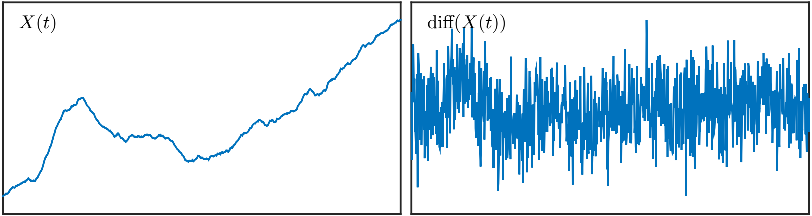

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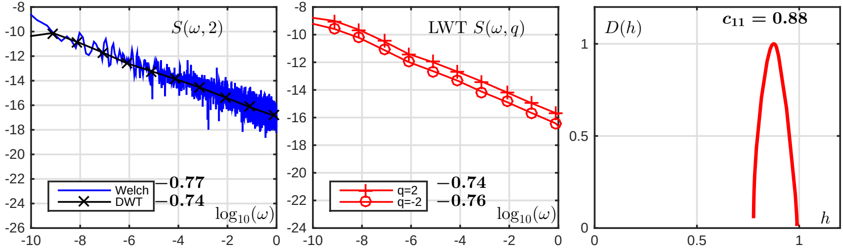

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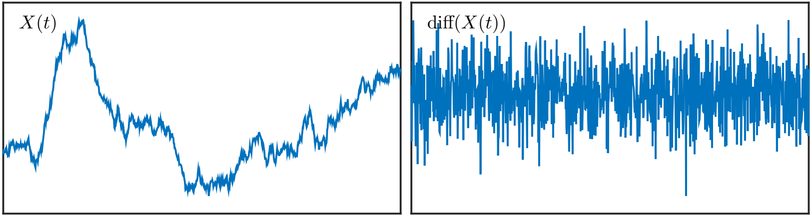

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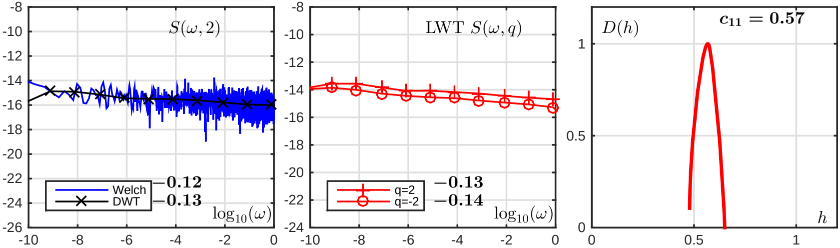

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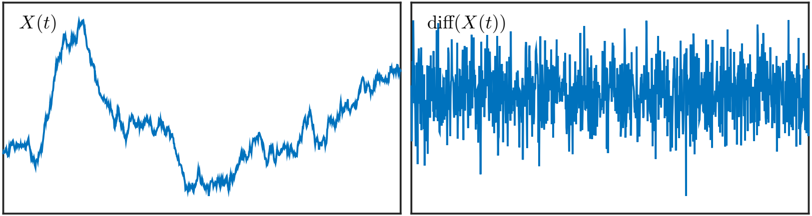
Intuitions: multifractal properties

Changing the regularity fluctuations $\sim c_2$



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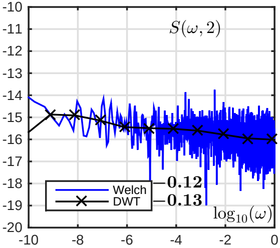

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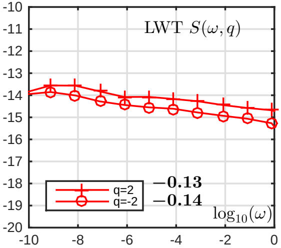

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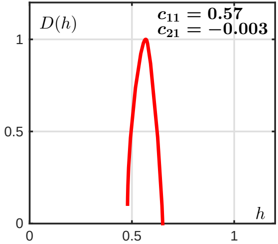

$\longrightarrow$ beyond 2nd order

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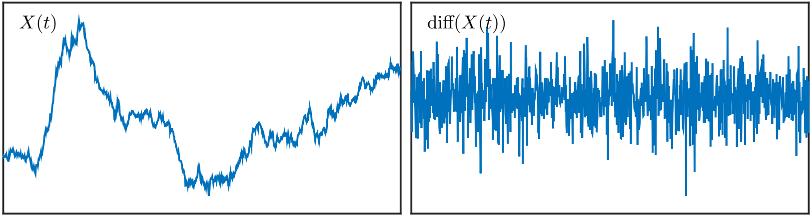
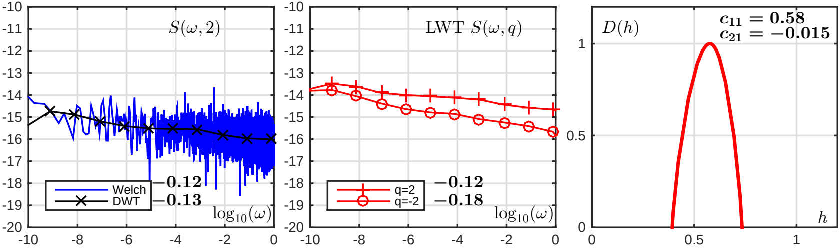


Spectrum, DWT ($q = 2$) Leaders ($q = \pm 2$) MF formalism



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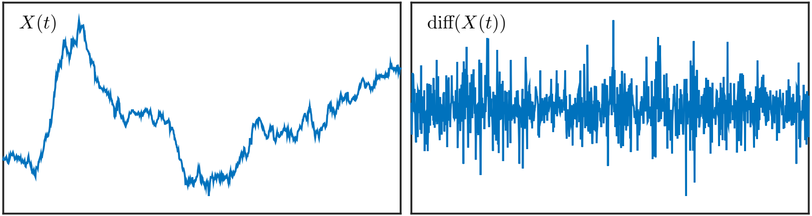

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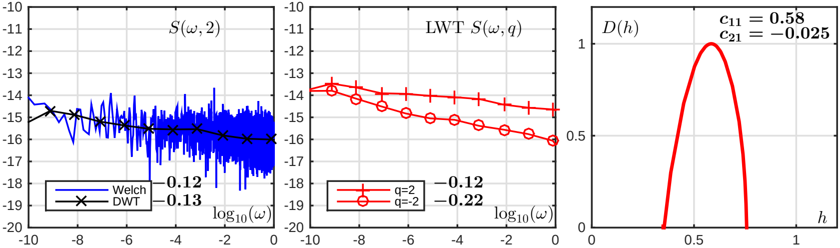

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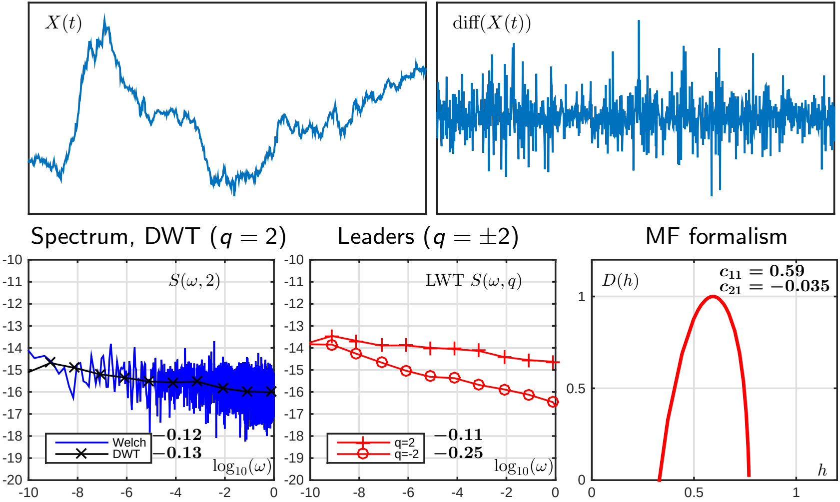

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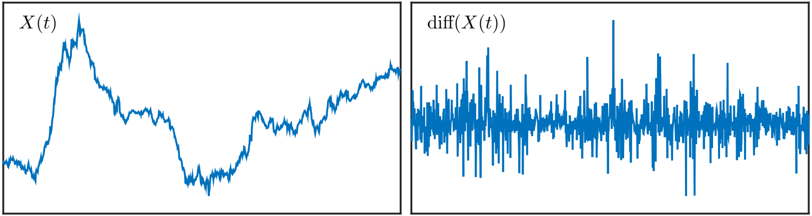
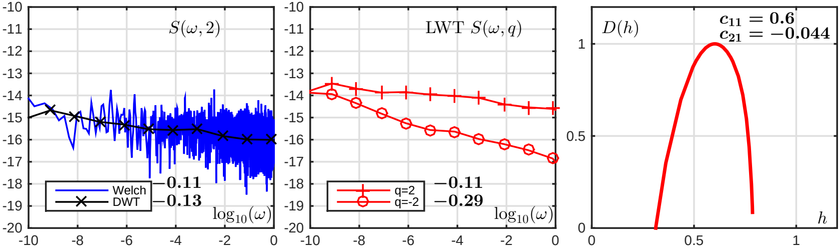


Spectrum, DWT ($q = 2$) | Leaders ($q = \pm 2$) | MF formalism



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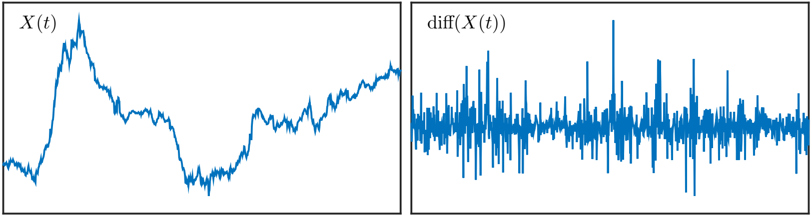

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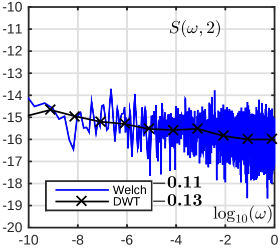

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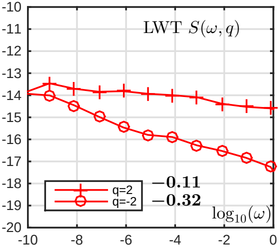

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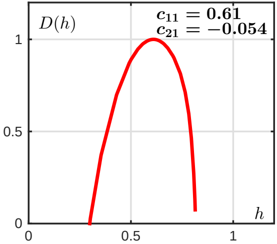

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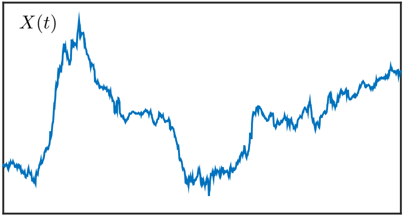

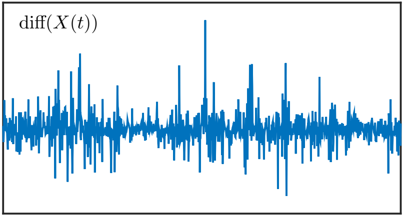

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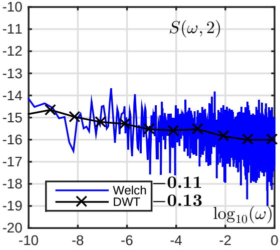

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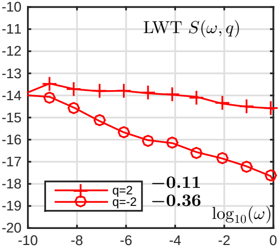

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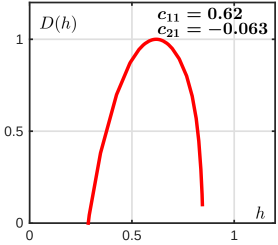

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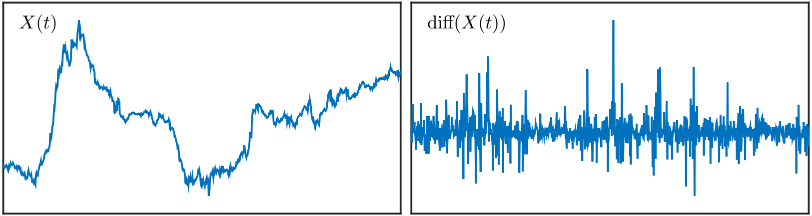

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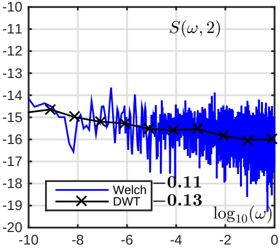

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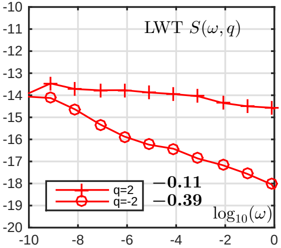

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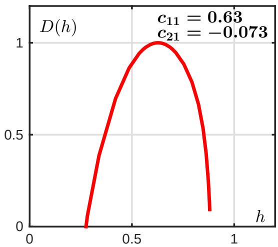

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Intuitions: multifractal properties

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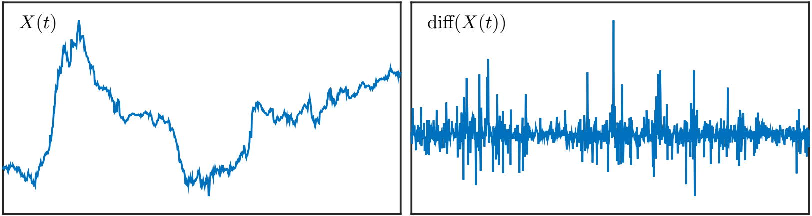
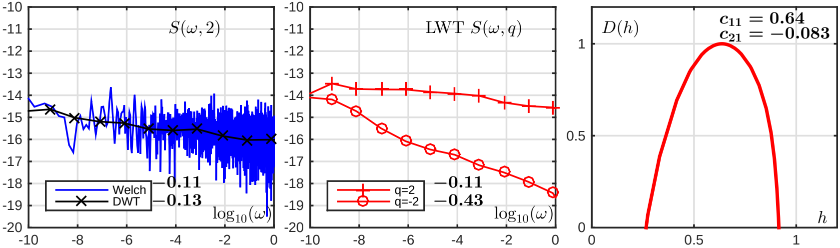


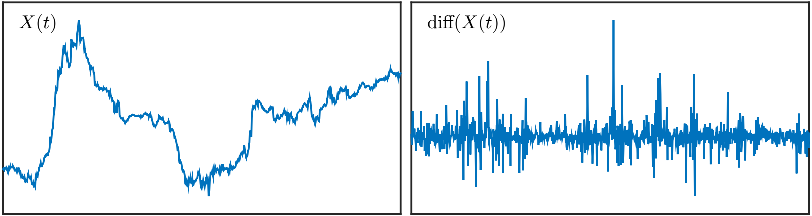
Spectrum, DWT ($q = 2$) Leaders ($q = \pm 2$) MF formalism



$\longrightarrow$ beyond 2nd order

Intuitions: multifractal properties

Changing the regularity fluctuations $\sim c_2$



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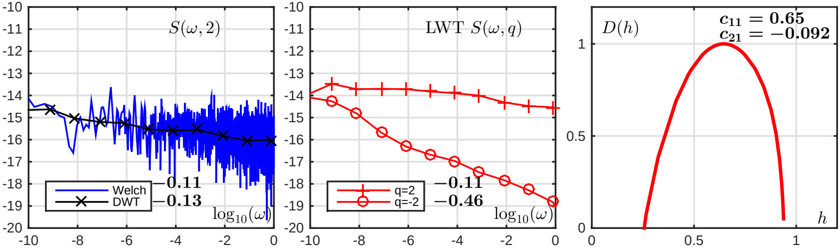

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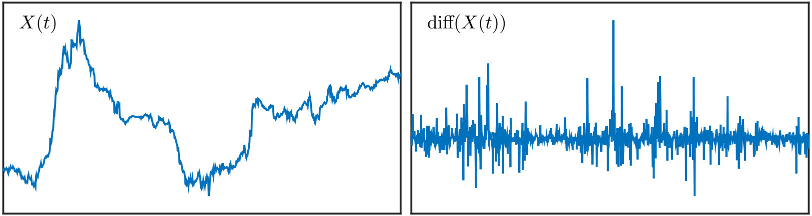

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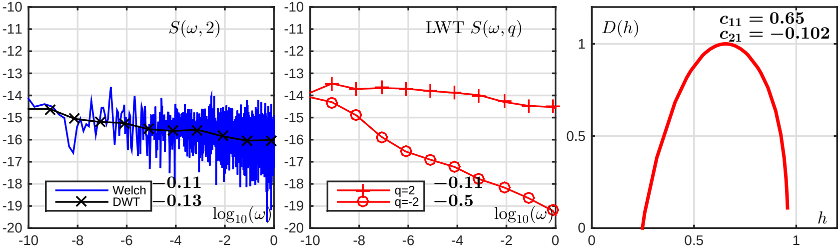


$\longrightarrow$ beyond 2nd order

Multifractal analysis: Applications

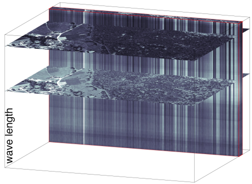
Successful use in large panel of applications of very different natures

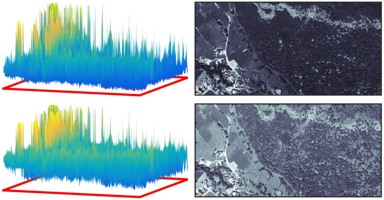
- physics (hydrodynamic turbulence, astrophysics and stellar plasmas, statistical physics, roughness of surfaces, . . .)
- biology (human heart rate variabilities, fMRI, physiological signals or images, . . .)
- geology (fault repartition)
- population geographical repartition, social behaviors
- computer network traffic
- finance and financial markets
- texture analysis
- Art investigation
- linguistic and text analysis
- . . .

Estimation of the multifractality parameter c_2

- Estimation of c_2 is challenging
  - linear regression based estimation $\boxed{\mathrm{Var}\,[\,\ln \ell(j,k)\,] = c_2^0 + c_2 \ln 2^j}$

Hyperspectral image [Sheeren'11]





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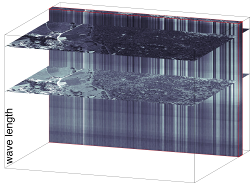
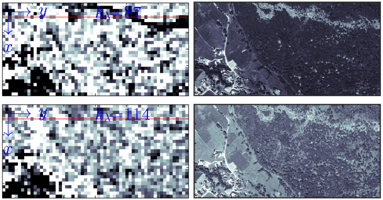


Goal: improve estimation



- c_{21} maps, patch size 64×64 ($= 4096$ pixels, 2 scales only)

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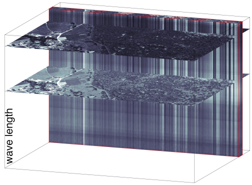

1. Bayesian estimation for c_2 for single image
   - robust semiparametric model for log-leaders $\longrightarrow$ Part 2
     [TIP15,ICASSP16]

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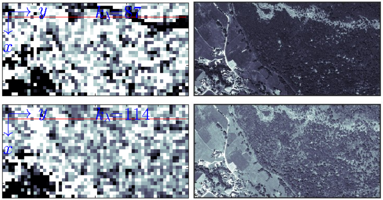

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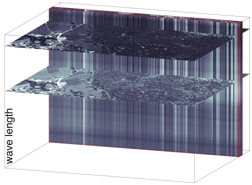

1. Bayesian estimation for c_2 for single image
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2. Bayesian estimation for c_2 for multivariate data
   - regularization using Markov field joint prior [EUSIPCO16,ICIP16,SIIMS18] $\longrightarrow$ Part 3

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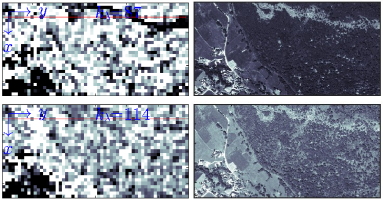

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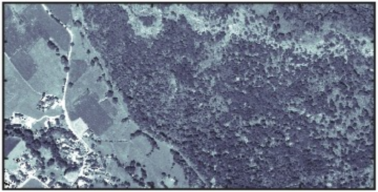
Part 1: Multifractal analysis
functional analysis, geometric description and higher-order statistics

Part 2: Bayesian model for single image
from Gaussian random field to data augmented model

Part 3: Bayesian model for multivariate data
Markov field joint prior for multifractal parameters



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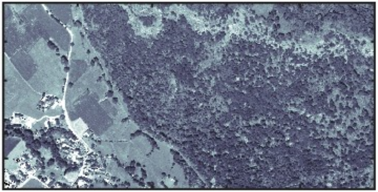
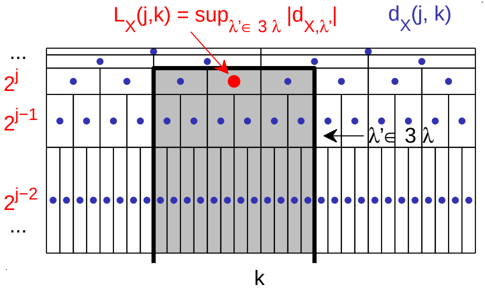


Gaussian random field model for log-leaders (1/2)

- Marginal distributions: log-Normal always good fit for multiscale histograms of multifractal cascades [Mandelbrot90]

$\longrightarrow$ log-leaders well approximated by Gaussian [ICASSP13,TIP15]

$$\boxed{l(j,k) = \ln \ell(j,k) \sim \mathcal{N}(\mathbb{E}[l(j,k)], \mathrm{Var}\,[l(j,k)])}$$



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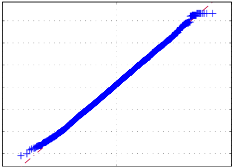
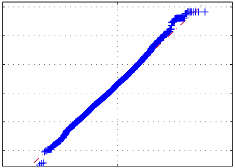
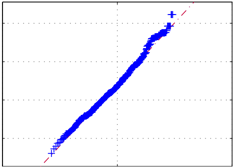


log-leaders $l(3,\cdot)$



log-leaders $l(4,\cdot)$



empirical marginals (qq-plots)

Multifractal random walk (MRW) [Bacry01,Robert10]

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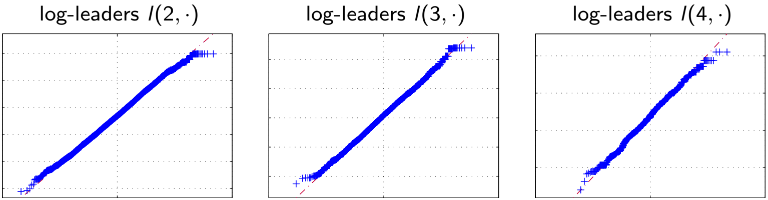

empirical marginals (qq-plots)

Log-Poisson Cascade [Mandelbrot]

Gaussian random field model for log-leaders (2/2)

- Mean $\boxed{\mathbb{E}[\ l(j,k)\ ] = c_1^0 + jc_1 \ln 2}$ (discarded below)
- Variance-covariance $\boxed{\mathrm{Var}[\ l(j,k)\ ] = c_2^0 + jc_2 \ln 2}$

  - asymptotic covariance decay: [Arneodo98]
    $\rightarrow$ linear in $\log(\Delta k)$
    $\rightarrow$ controlled by (c_2, c_2^0)

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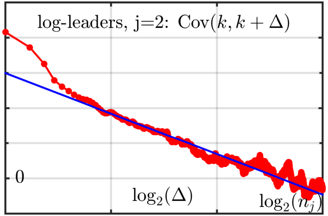

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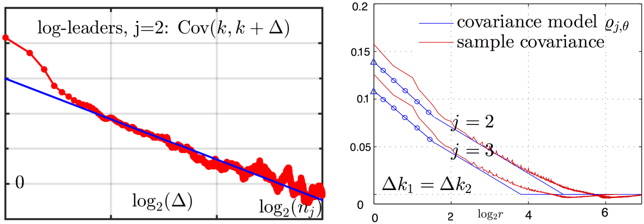


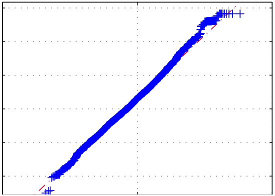
$\rightarrow$ piecewise logarithmic model $\varrho_{j,(c_2,c_2^0)}(\Delta k)$ [ICASSP15,TIP15]

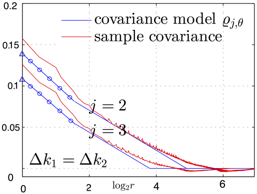
with parameters (c_2, c_2^0) $\longrightarrow$ Covariance matrix $\Sigma_{j,(c_2,c_2^0)}$

From a standard likelihood w.r.t. (c_2, c_2^0) ...

– log-leaders at scale j: $\boldsymbol{l}_j \triangleq (l(j,1), l(j,2), \dots)$

$$p(\boldsymbol{l}_j | (c_2, c_2^0)) \propto (\det \Sigma_{j,(c_2,c_2^0)})^{-\frac{1}{2}} \exp\left(-(\boldsymbol{l}_j^T \Sigma_{j,(c_2,c_2^0)}^{-1} \boldsymbol{l}_j)/2\right)$$





empirical marginals (qq-plot) and covariance

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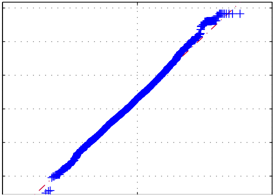
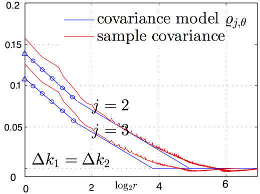
– inter-scale independence assumption: $\boldsymbol{l} = [\boldsymbol{l}_{j_1}^T, \dots, \boldsymbol{l}_{j_1}^T]^T$

$$p(\boldsymbol{l}|(c_2, c_2^0)) \propto \prod_{j=j_1}^{j_2} p(\boldsymbol{l}_j|(c_2, c_2^0))$$





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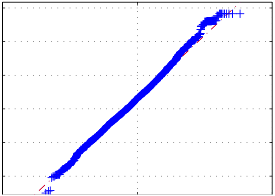

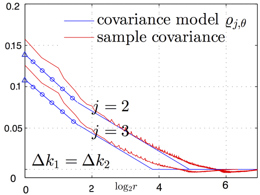

empirical marginals (qq-plot) and covariance

✗ inversion of $\Sigma_{j,(c_2,c_2^0)}$ prohibitive	→ Whittle approximation
✗ constraints: $\Sigma_{j,(c_2,c_2^0)}$ p.d.	→ reparametrization
✗ conjugacy of priors for parameters	→ data augmentation

... to a data augmented likelihood [TIP15,ICASSP16]

1. Whittle approximation $\Longrightarrow$ Fourier transform (DFT) of centered log-leaders $\boldsymbol{l}_j$

$$\boldsymbol{y}_j = DFT(\boldsymbol{l}_j) \quad \longrightarrow \quad p(\boldsymbol{l}_j|(c_2, c_2^0)) \propto |\boldsymbol{f}_{j,(c_2,c_2^0)}^{-1}| \exp\left(\frac{\boldsymbol{y}_j^* \boldsymbol{y}_j}{\boldsymbol{f}_{j,(c_2,c_2^0)}}\right)$$

$$\longrightarrow \quad \boldsymbol{F}_{(c_2,c_2^0)} = \mathrm{diag}(\boldsymbol{f}_{j_1,(c_2,c_2^0)}, \ldots, \boldsymbol{f}_{j_2,(c_2,c_2^0)})$$

$\boldsymbol{f}_{j,(c_2,c_2^0)}$: spectrum associated with model $\varrho_{j,(c_2,c_2^0)}(\Delta k)$

$p(\boldsymbol{l}|(c_2, c_2^0))$

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X - $\boldsymbol{F}_{(c_2,c_2^0)} > 0$: joint constraints on parameters

---

$p(\boldsymbol{l} | (c_2, c_2^0))$

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X - $\boldsymbol{F}_{(c_2,c_2^0)} > 0$: joint constraints on parameters

2. Reparametrization $\Longrightarrow$ independent positivity constraints on parameters

$$\boldsymbol{c_2} = (c_{20}, c_{21}) \triangleq \psi((c_2, c_2^0)) \in \mathbb{R}_\star^{+2} \quad \longrightarrow \quad \textit{separable } \boldsymbol{F}_{(c_2,c_2^0)} = c_{20}\boldsymbol{F}_0 + c_{21}\boldsymbol{F}_1$$

$\boldsymbol{F}_0$, $\boldsymbol{F}_1$ diagonal, positive definite, known and fixed

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$$p(\boldsymbol{l}|(c_2, c_2^0)) \quad \longrightarrow \quad p(\boldsymbol{l}|\boldsymbol{c_2})$$

. . . to a data augmented likelihood

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X - conjugacy of priors for $\boldsymbol{c_2}$: LH does not factor in c_{20} and c_{21}

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$\boldsymbol{F}_0$, $\boldsymbol{F}_1$ diagonal, positive definite, known and fixed

X - conjugacy of priors for $\boldsymbol{c_2}$: LH does not factor in c_{20} and c_{21}

3. Data augmentation $\Longrightarrow$ introduce hidden mean $\boldsymbol{\mu}_j$ for $\boldsymbol{y}_j$ [Tanner'87, van Dyk'01]

$\Longrightarrow$ complex Gaussian model for DFT $\boldsymbol{y} = [\boldsymbol{y}_{j_1}^T, \ldots, \boldsymbol{y}_{j_2}^T]^T$ of log-leaders

$$\begin{cases} \boldsymbol{y}|\boldsymbol{\mu}, c_{20} & \sim \mathcal{CN}(\boldsymbol{\mu}, c_{20}\boldsymbol{F}_0) \quad \text{observed data} \\ \boldsymbol{\mu}|c_{21} & \sim \mathcal{CN}(\boldsymbol{0}, c_{21}\boldsymbol{F}_1) \quad \text{hidden mean} \end{cases}$$

$$p(\boldsymbol{l}|(c_2, c_2^0)) \quad \longrightarrow \quad p(\boldsymbol{l}|\boldsymbol{c_2}) \quad \longrightarrow \quad p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{c_2}) \propto p(\boldsymbol{y}|\boldsymbol{\mu}, c_{20})\, p(\boldsymbol{\mu}|c_{21})$$

Augmented likelihood based Bayesian model

[ICASSP16]

- Augmented likelihood w.r.t. $\boldsymbol{c}_2 = \psi((c_2, c_2^0))$

$$p(\boldsymbol{y}, \boldsymbol{\mu} | \boldsymbol{c}_2) \propto {c_{20}}^{-N_Y} \exp\left(-\frac{1}{c_{20}} (\boldsymbol{y} - \boldsymbol{\mu})^H \boldsymbol{F}_0^{-1} (\boldsymbol{y} - \boldsymbol{\mu}) \right) \times {c_{21}}^{-N_Y} \exp\left(-\frac{1}{c_{21}} \boldsymbol{\mu}^H \boldsymbol{F}_1^{-1} \boldsymbol{\mu} \right)$$

- Prior distribution for parameters

c_{2i} as variance of Gaussian

$\rightarrow$ inverse-gamma prior $c_{2i} \sim \mathcal{IG}(\alpha_i, \beta_i)$ is conjugate

- Posterior distribution

$$p(\boldsymbol{c}_2, \boldsymbol{\mu} | \boldsymbol{y}) \propto p(\boldsymbol{y}, \boldsymbol{\mu} | \boldsymbol{c}_2) p(c_{20}) p(c_{21})$$

- Bayesian estimators

$\rightarrow$ marginal posterior mean estimator (MMSE) ${\boldsymbol{c}_2}^{\text{MMSE}} = \mathbb{E}[\boldsymbol{c}_2 | \boldsymbol{y}]$

- Gibbs sampler

$p(\boldsymbol{\mu} | \boldsymbol{c}_2, \boldsymbol{y})$ closed-form Gaussian distribution

$p(c_{2i} | c_{2i' \neq i}, \boldsymbol{\mu}, \boldsymbol{y})$ closed-form inverse-gamma distributions

all standard distributions $\rightarrow$ no Metropolis-Hasting moves

summary detailed

Augmented likelihood based Bayesian model

[ICASSP16]

- Augmented likelihood w.r.t. $\boldsymbol{c_2} = \psi((c_2, c_2^0))$

$$p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{c_2}) \propto {c_{20}}^{-N_Y} \exp\left(-\frac{1}{c_{20}}(\boldsymbol{y}-\boldsymbol{\mu})^H \boldsymbol{F}_0^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right) \times {c_{21}}^{-N_Y} \exp\left(-\frac{1}{c_{21}}\boldsymbol{\mu}^H \boldsymbol{F}_1^{-1}\boldsymbol{\mu}\right)$$

- Prior distribution for parameters

c_{2i} as variance of Gaussian

$\rightarrow$ inverse-gamma prior $c_{2i} \sim \mathcal{IG}(\alpha_i, \beta_i)$ is conjugate

- Posterior distribution

$$p(\boldsymbol{c_2}, \boldsymbol{\mu}|\boldsymbol{y}) \propto p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{c_2})p(c_{20})p(c_{21})$$

- Bayesian estimators

$\rightarrow$ marginal posterior mean estimator (MMSE) ${\boldsymbol{c_2}}^{\text{MMSE}} = \mathbb{E}[\boldsymbol{c_2}|\boldsymbol{y}]$

- Gibbs sampler

$p(\boldsymbol{\mu}|\boldsymbol{c_2}, \boldsymbol{y})$ closed-form Gaussian distribution

$p(c_{2i}|c_{2i'\neq i}, \boldsymbol{\mu}, \boldsymbol{y})$ closed-form inverse-gamma distributions

all standard distributions $\rightarrow$ no Metropolis-Hasting moves

summary detailed

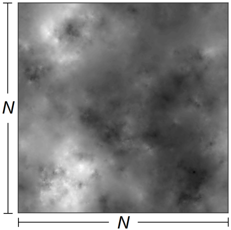
Does it work?

- Wavelet transform → Daubechies' mother wavelet ($N_\psi = 2$)
- Sample sizes and analysis scales

2D multifractal Random walk (MRW)

N	2^6	2^7	2^8	2^9	2^{10}	2^{11}
j_1	1	1	2	2	2	2
j_2	2	3	4	5	6	7

small large



- Prior specification → non informative priors

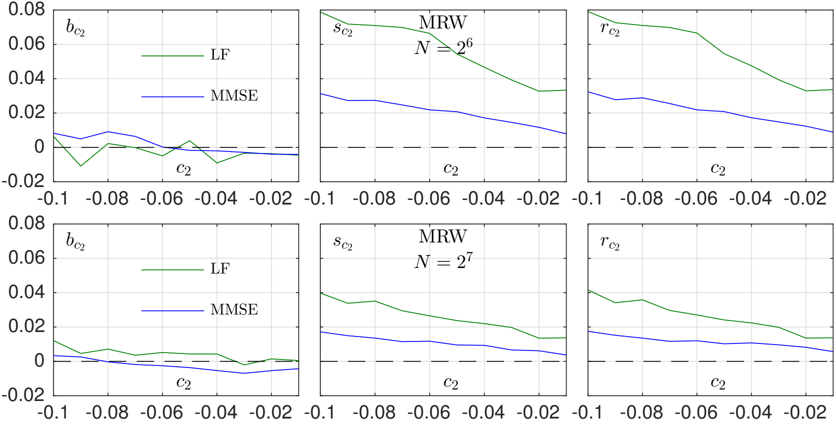
– $c_{2i} \sim \mathcal{IG}(\alpha_i, \beta_i)$ with $(\alpha_i, \beta_i) = (10^{-3}, 10^{-3}) \sim$ Jeffreys' prior

- Estimation performance

MEAN	BIAS	STD	RMSE
$\mathrm{m}_{c_{2i}} = \widehat{\mathbb{E}}[\hat{c}_{2i}]$	$\mathrm{b}_{c_{2i}} = \mathrm{m}_{c_{2i}} - c_{2i}$	$\mathrm{s}_{c_{2i}} = \sqrt{\widehat{\mathrm{Var}}[\hat{c}_{2i}]}$	$\mathrm{r}_{c_{2i}} = \sqrt{\mathrm{b}_{c_{2i}}^2 + \mathrm{s}_{c_{2i}}^2}$

→ assessed on 100 independent realizations

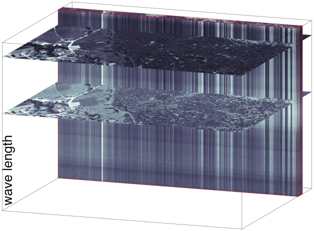
Estimation performance for c_{21}: 2D MRW

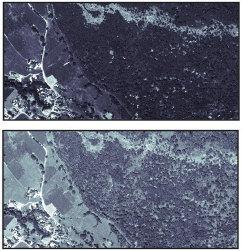


Estimation performance for c_{21} – 2D MRW – N=$2^6, 2^7$

- BIAS — Bayesian estimator $\sim$ LF estimator
- STD, RMSE — Bayesian estimator $\sim 1.5 - 4$ times below LF estimator
- *less than* 5 *times slower than LF*

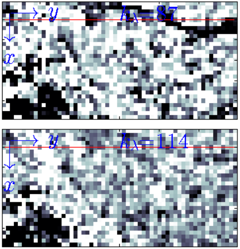
Empirical data: hyperspectral image



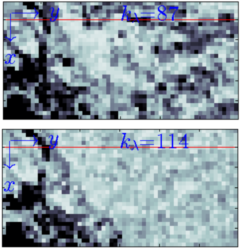


- c_{21} maps

linear regression:



Bayesian:



- $n = 64 \times 64$ patches

$\rightarrow$ spatial coherence

Part 1: Multifractal analysis

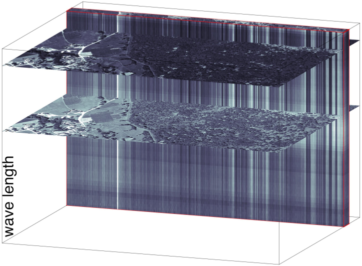
functional analysis, geometric description and higher-order statistics

Part 2: Bayesian model for single image

from Gaussian random field to data augmented model

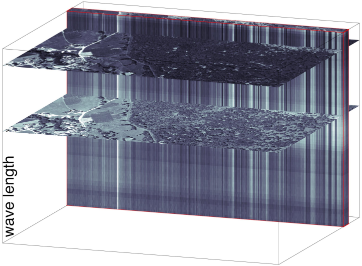
Part 3: Bayesian model for multivariate data

Markov field joint prior for multifractal parameters



Part 3: Bayesian model for multivariate data

Markov field joint prior for multifractal parameters



Estimation for multifractality parameter c_2

- Multifractal formalism: linear regressions

  √ successful for standard situations

  - single data
  - homogeneous
  - sufficient length

  X small size piece of data

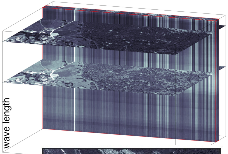
- Generic statistical model for log-leaders (Part 2)

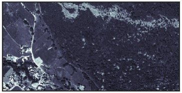
  $\longrightarrow$ additional model assumptions for single data

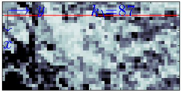
  √ improved estimation performance $\rightarrow$ small size

  √ Bayesian model and estimators

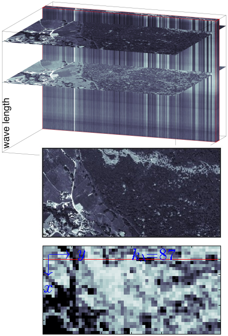
  √ fast estimation algorithm







Estimation for multifractality parameter c_2



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  √ fast estimation algorithm

- Many data components: multivariate / joint estimation?

  $\longrightarrow$ organization of data components: prior information, regularization

  $\longrightarrow$ improve estimation

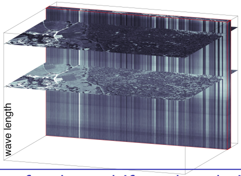
Multifractal analysis for multivariate images?

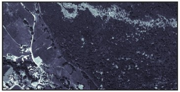
- Motivations
  - naturally multivariate data
  - $\longrightarrow$ multi-temporal, multi-band, multi-modal, voxels,...
  - non-homogeneous data
  - $\longrightarrow$ localize: collection of small homogeneous pieces

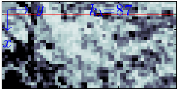
$\rightarrow$ joint analysis of the whole dataset? [Prats-Montalban'11]

- Multifractal analysis of multivariate images? [Meneveau'90,HW.etal'18]
  - intrinsically univariate definition of the multifractal spectrum $D(h)$
  - X multivariate spectrum: conceptual limit $\sim$pair of time series / images

$\rightarrow$ **joint estimation of multifractal parameters**







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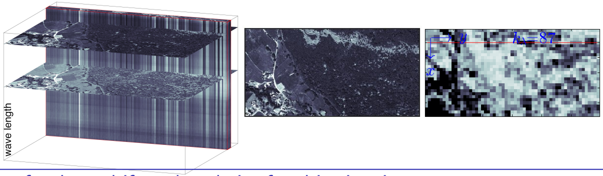

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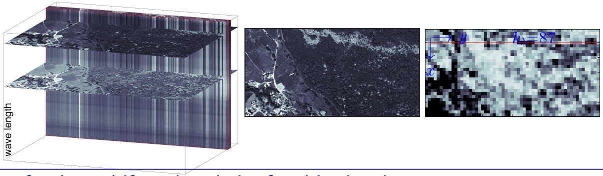
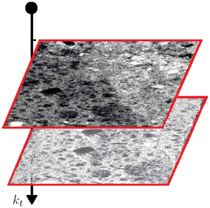
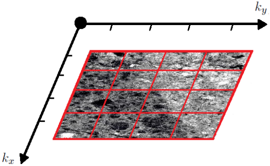


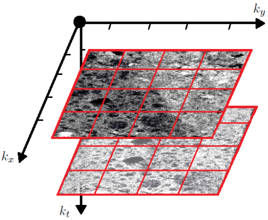
Images: multivariate scenarios



1D



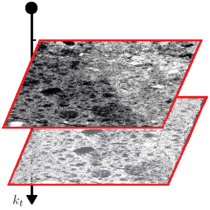
2D



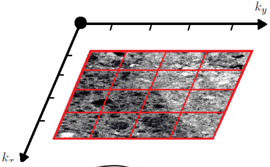
3D

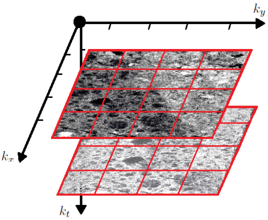
- 1D $\rightarrow$ temporal/spectral sequence of images
- 2D $\rightarrow$ spatial patches (e.g., patched-based local multifractal analysis)
- 3D $\rightarrow$ spatio-temporal/spectral patches

Images: multivariate scenarios



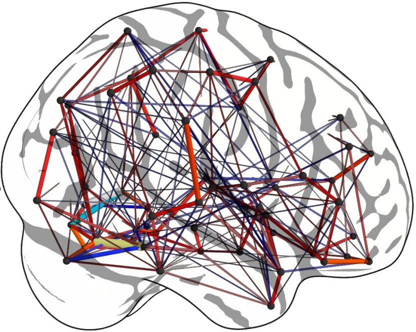
1D





3D

- 1D → temporal
- 2D → spatial p ifractal analysis)
- 3D → spatio-te



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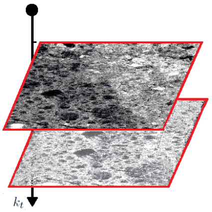
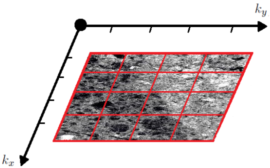
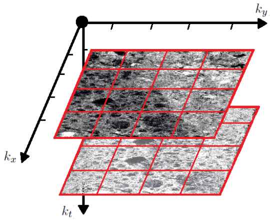


1D



2D



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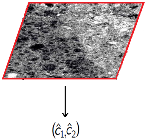
Strategy: hierarchical Bayesian model

	single data
data	y
param.	θ
LH	$p(y\|\theta)$
prior	$p(\theta)$
posterior	$p(\theta\|y) \propto p(y\|\theta)p(\theta)$

independent/univariate priors



Strategy: hierarchical Bayesian model

	single data $\longrightarrow$	**multivariate data**
data	y	$y_1, \quad y_2, \quad \dots \quad , \quad y_K$
param.	θ	$\theta_1, \quad \theta_2, \quad \dots \quad , \quad \theta_K$
LH	$p(y\|\theta)$	$p(y_1\|\theta_1), p(y_2\|\theta_2), \dots, p(y_K\|\theta_K)$
prior	$p(\theta)$	$p(\theta_1), \quad p(\theta_2), \dots, \quad p(\theta_K)$
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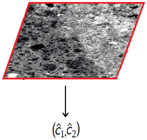

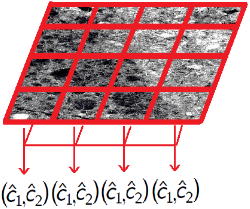

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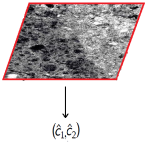
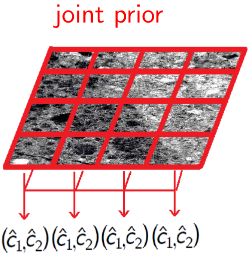
posterior	$p(\theta\|y) \propto p(y\|\theta)p(\theta)$		





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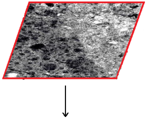
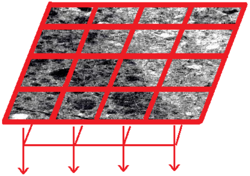
posterior	$p(\theta\|y) \propto p(y\|\theta)p(\theta)$	$p(\theta_1, \theta_2, \ldots, \theta_K\|y_1, y_2, \ldots, y_K, \beta)$
		$\propto \left(\prod_{k=1}^{K} p(y_k\|\theta_k)\right) p(\theta_1, \theta_2, \ldots, \theta_K, \beta)$

joint prior


$(\hat{c}_1,\hat{c}_2)$


$(\hat{c}_1,\hat{c}_2)(\hat{c}_1,\hat{c}_2)(\hat{c}_1,\hat{c}_2)(\hat{c}_1,\hat{c}_2)$

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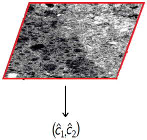
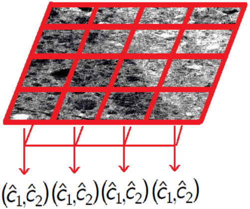
		$\propto \left(\prod_{k=1}^{K} p(y_k\|\theta_k)\right) p(\theta_1, \theta_2, \ldots, \theta_K\|\beta)p(\beta)$

joint prior hyper-prior

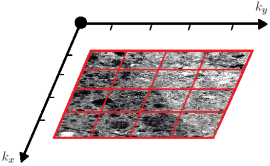




Strategy: hierarchical Bayesian model

For decomposition of image $\mathbf{X}$ into spatial patches $\mathbf{X}_{\boldsymbol{k}}$

1. Statistical model $p(\mathbf{y}_{\boldsymbol{k}}, \boldsymbol{\mu}_{\boldsymbol{k}} | \boldsymbol{c}_{2\boldsymbol{k}})$
   - $\boldsymbol{c}_{2\boldsymbol{k}} = [c_{20,\boldsymbol{k}}, c_{21,\boldsymbol{k}}]^T$
   - $\mathbf{y}_{\boldsymbol{k}}$ Fourier coefficients
   - $\boldsymbol{\mu}_{\boldsymbol{k}}$ hidden mean



Patches $\mathbf{X}_{\boldsymbol{k}}$,
$\boldsymbol{k} = (m_1, m_2)$
$m_1 = 1, \ldots, N_x$
$m_2 = 1, \ldots, N_y$

2. Prior independence between patches: LH

$$p(\mathbf{Y}, \mathbf{M} | \mathbf{C}_2) \propto \prod_{\boldsymbol{k}} p(\mathbf{y}_{\boldsymbol{k}}, \boldsymbol{\mu}_{\boldsymbol{k}} | \boldsymbol{c}_{2\boldsymbol{k}})$$

   - $\mathbf{C}_2 = \{\boldsymbol{c}_{20}, \boldsymbol{c}_{21}\}$ with $\boldsymbol{c}_{2i} = \{c_{2i,\boldsymbol{k}}\}_{\boldsymbol{k}}$
   - $\mathbf{Y} = \{\mathbf{y}_{\boldsymbol{k}}\}_{\boldsymbol{k}}$
   - $\mathbf{M} = \{\boldsymbol{\mu}_{\boldsymbol{k}}\}_{\boldsymbol{k}}$

- Posterior

independent/univariate priors

$$p(\mathbf{C}_2, \mathbf{Z}, \mathbf{M} | \mathbf{Y}, \rho) \propto \underbrace{p(\mathbf{Y}, \mathbf{M} | \mathbf{C}_2)}_{\text{augmented likelihood}} \times \prod_{i=0}^{1} \underbrace{\prod_{\boldsymbol{k}} p(c_{2i,\boldsymbol{k}} | \alpha, \beta)}_{\text{independent } \mathcal{IG} \text{ priors}}$$

Strategy: hierarchical Bayesian model

For decomposition of image $\mathbf{X}$ into spatial patches $\mathbf{X}_{\boldsymbol{k}}$

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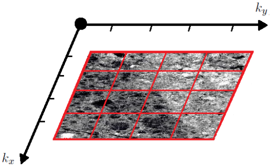

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$\boldsymbol{k} = (m_1, m_2)$
$m_1 = 1, \ldots, N_x$
$m_2 = 1, \ldots, N_y$

- Posterior → design of regularizing joint priors for $\mathbf{C}_2$

$$p(\mathbf{C}_2, \mathbf{Z}, \mathbf{M} | \mathbf{Y}, \rho) \propto \underbrace{p(\mathbf{Y}, \mathbf{M} | \mathbf{C}_2)}_{\text{augmented likelihood}} \times \prod_{i=0}^{1} \underbrace{\prod_{\boldsymbol{k}} p(c_{2i,\boldsymbol{k}} | \alpha, \beta)}_{\text{independent } \mathcal{IG} \text{ priors}}$$

Gamma Markov random field (GaMRF) prior for $\mathbf{C_2}$

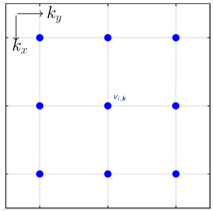
- Posterior independent/univariate priors

$$p(\mathbf{C_2}, \mathbf{Z}, \mathbf{M} | \mathbf{Y}, \rho) \propto \underbrace{p(\mathbf{Y}, \mathbf{M} | \mathbf{C_2})}_{\text{augmented likelihood}} \quad \prod_{i=0}^{1} \underbrace{\prod_{\boldsymbol{k}} p(c_{2i,\boldsymbol{k}} | \alpha, \beta)}_{\text{independent } \mathcal{IG} \text{ priors}}$$

- Regularization $\rightarrow$ specify $(\alpha, \beta) = (\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}})$ within a hidden GaMRF

- Joint GaMRF prior for $\mathbf{C_2}_i$, $i = 0, 1$

  $\rightarrow$ enforces smooth evolution of variances of Gaussian [Dikmen'10]



Gamma Markov random field (GaMRF) prior for $\mathbf{C_2}$

- Posterior

independent/univariate priors

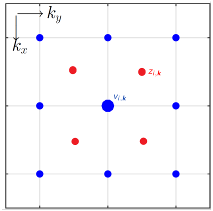
$$p(\mathbf{C_2}, \mathbf{Z}, \mathbf{M}|\mathbf{Y}, \rho) \propto \underbrace{p(\mathbf{Y}, \mathbf{M}|\mathbf{C_2})}_{\text{augmented likelihood}} \prod_{i=0}^{1} \underbrace{\prod_{\boldsymbol{k}} p(c_{2i,\boldsymbol{k}}|\alpha, \beta)}_{\text{independent } \mathcal{IG} \text{ priors}}$$

- Regularization $\rightarrow$ specify $(\alpha, \beta) = (\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}})$ within a hidden GaMRF

- Joint GaMRF prior for $\mathbf{C_2}_i$, $i = 0, 1$

  $\rightarrow$ enforces smooth evolution of variances of Gaussian [Dikmen'10]

- positive auxiliary variables $\boldsymbol{z}_i = \{z_{i,\boldsymbol{k}}\}_{\boldsymbol{k}}$, $i = 0, 1$



Gamma Markov random field (GaMRF) prior for $\mathbf{C_2}$

- Posterior

independent/univariate priors

$$p(\mathbf{C_2}, \mathbf{Z}, \mathbf{M}|\mathbf{Y}, \rho) \propto \underbrace{p(\mathbf{Y}, \mathbf{M}|\mathbf{C_2})}_{\text{augmented likelihood}} \prod_{i=0}^{1} \underbrace{\prod_{\boldsymbol{k}} p(c_{2i,\boldsymbol{k}}|\alpha, \beta)}_{\text{independent } \mathcal{IG} \text{ priors}}$$

- Regularization $\rightarrow$ specify $(\alpha, \beta) = (\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}})$ within a hidden GaMRF

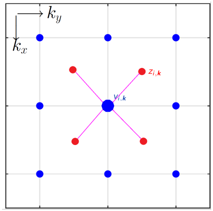
- Joint GaMRF prior for $\mathbf{C_2}_i$, $i = 0, 1$

  $\rightarrow$ enforces smooth evolution of variances of Gaussian [Dikmen'10]

- positive auxiliary variables $\boldsymbol{z}_i = \{z_{i,\boldsymbol{k}}\}_{\boldsymbol{k}}$, $i = 0, 1$
- each $c_{2i,\boldsymbol{k}}$ is connected to 4 variables $z_{i,\boldsymbol{k}'}$

  $$\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k}) = \{(m_1, m_2) + (l_x, l_y)\}_{l_x, l_y = 0,1}$$

  via edges with weights ρ_i



Gamma Markov random field (GaMRF) prior for $\mathbf{C}_2$

- Posterior — independent/univariate priors

$$p(\mathbf{C_2}, \mathbf{Z}, \mathbf{M}|\mathbf{Y}, \rho) \propto \underbrace{p(\mathbf{Y}, \mathbf{M}|\mathbf{C_2})}_{\text{augmented likelihood}} \prod_{i=0}^{1} \underbrace{\prod_{\boldsymbol{k}} p(c_{2\,i,\boldsymbol{k}}|\alpha, \beta)}_{\text{independent } \mathcal{IG} \text{ priors}}$$

- Regularization $\rightarrow$ specify $(\alpha, \beta) = (\alpha_{\boldsymbol{k}}, \beta_{\boldsymbol{k}})$ within a hidden GaMRF

- Joint GaMRF prior for $\mathbf{C_2}_i$, $i = 0, 1$

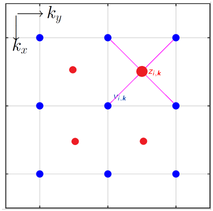
  $\rightarrow$ enforces smooth evolution of variances of Gaussian [Dikmen'10]

- positive auxiliary variables $\boldsymbol{z}_i = \{z_{i,\boldsymbol{k}}\}_{\boldsymbol{k}}$, $i = 0, 1$
- each $c_{2\,i,\boldsymbol{k}}$ is connected to 4 variables $z_{i,\boldsymbol{k}'}$

  $$\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k}) = \{(m_1, m_2) + (l_x, l_y)\}_{l_x, l_y = 0,1}$$

  via edges with weights ρ_i
- each $z_{i,k}$ is connected to 4 variables $c_{2\,i,\boldsymbol{k}'}$

  $$\boldsymbol{k}' \in \mathcal{V}_z(\boldsymbol{k}) = \{(m_1, m_2) + (l_x, l_y)\}_{l_x, l_y = -1,0}$$



GaMRF prior for $\mathbf{C_2}$: density and conditionals

$$p(\mathbf{C_2}_i, \boldsymbol{Z}_i|\rho_i) = \frac{1}{K(\rho_i)} \prod_{\boldsymbol{k}} e^{-(4\rho_i+1)\ln c_{2i,\boldsymbol{k}}} e^{(4\rho_i-1)\ln z_{i,k}} \times e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}}\sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$= \frac{1}{K(\rho_i)} \prod_{\boldsymbol{k}} {c_{2i,\boldsymbol{k}}}^{-(4\rho_i+1)} {z_{i,k}}^{(4\rho_i-1)} \times e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}}\sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

- Conditionals for $c_{2i,\boldsymbol{k}}$: $\rightarrow$ inverse-gamma

$$p(c_{2i,\boldsymbol{k}}|\boldsymbol{Z}_i, \rho_i) \propto {c_{2i,\boldsymbol{k}}}^{-(4\rho_i+1)} e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}}\sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$
$$\sim \mathcal{IG}\Big(4\rho_i, \rho_i \sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,\boldsymbol{k}'}\Big)$$

- Conditionals for $z_{i,\boldsymbol{k}}$: $\rightarrow$ gamma

$$p(z_{i,\boldsymbol{k}}|\mathbf{C_2}_i, \rho_i) \propto {z_{i,k}}^{(4\rho_i-1)} \prod_{\boldsymbol{k}} e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}}\sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$
$$\propto {z_{i,k}}^{(4\rho_i-1)} e^{-\sum_{\boldsymbol{k}} \frac{\rho_i}{c_{2i,\boldsymbol{k}}}\sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$
$$\propto {z_{i,k}}^{(4\rho_i-1)} e^{-\rho_i z_{i,k}\sum_{\boldsymbol{k}''\in\mathcal{V}_z(\boldsymbol{k})} \frac{1}{c_{2i,\boldsymbol{k}''}}}$$
$$\sim \mathcal{G}\Big(4\rho_i, \rho_i \sum_{\boldsymbol{k}''\in\mathcal{V}_z(\boldsymbol{k})} c_{2\,i,\boldsymbol{k}''}^{-1}\Big)$$

GaMRF prior for $\mathbf{C_2}$: density and conditionals

$$p(\mathbf{C_2}_i, \boldsymbol{Z}_i|\rho_i) = \frac{1}{K(\rho_i)} \prod_{\boldsymbol{k}} e^{-(4\rho_i+1)\ln c_{2i,\boldsymbol{k}}} e^{(4\rho_i-1)\ln z_{i,k}} \times e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$= \frac{1}{K(\rho_i)} \prod_{\boldsymbol{k}} c_{2i,\boldsymbol{k}}{}^{-(4\rho_i+1)} z_{i,k}{}^{(4\rho_i-1)} \times e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

- Conditionals for $c_{2i,\boldsymbol{k}}$: $\rightarrow$ inverse-gamma

$$p(c_{2i,\boldsymbol{k}}|\boldsymbol{Z}_i,\rho_i) \propto c_{2i,\boldsymbol{k}}{}^{-(4\rho_i+1)} e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$\sim \mathcal{IG}\left(4\rho_i, \rho_i \sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,\boldsymbol{k}'}\right)$$

- Conditionals for $z_{i,\boldsymbol{k}}$: $\rightarrow$ gamma

$$p(z_{i,\boldsymbol{k}}|\mathbf{C_2}_i,\rho_i) \propto z_{i,k}{}^{(4\rho_i-1)} \prod_{\boldsymbol{k}} e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$\propto z_{i,k}{}^{(4\rho_i-1)} e^{-\sum_{\boldsymbol{k}} \frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$\propto z_{i,k}{}^{(4\rho_i-1)} e^{-\rho_i z_{i,k} \sum_{\boldsymbol{k}''\in\mathcal{V}_z(\boldsymbol{k})} \frac{1}{c_{2i,\boldsymbol{k}''}}}$$

$$\sim \mathcal{G}\left(4\rho_i, \rho_i \sum_{\boldsymbol{k}''\in\mathcal{V}_z(\boldsymbol{k})} c_{2}{}^{-1}_{i,\boldsymbol{k}''}\right)$$

GaMRF prior for $\mathbf{C_2}$: density and conditionals

$$p(\mathbf{C_2}_i, \boldsymbol{Z}_i|\rho_i) = \frac{1}{K(\rho_i)} \prod_{\boldsymbol{k}} e^{-(4\rho_i+1)\ln c_{2i,\boldsymbol{k}}} e^{(4\rho_i-1)\ln z_{i,k}} \times e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$= \frac{1}{K(\rho_i)} \prod_{\boldsymbol{k}} {c_{2i,\boldsymbol{k}}}^{-(4\rho_i+1)} {z_{i,k}}^{(4\rho_i-1)} \times e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

- Conditionals for $c_{2i,\boldsymbol{k}}$: → inverse-gamma

$$p(c_{2i,\boldsymbol{k}}|\boldsymbol{Z}_i, \rho_i) \propto {c_{2i,\boldsymbol{k}}}^{-(4\rho_i+1)} e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$\sim \mathcal{IG}(4\rho_i, \rho_i \textstyle\sum_{\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k})} z_{i,\boldsymbol{k}'})$$

- Conditionals for $z_{i,\boldsymbol{k}}$: → gamma

$$p(z_{i,\boldsymbol{k}}|\mathbf{C_2}_i, \rho_i) \propto {z_{i,k}}^{(4\rho_i-1)} \prod_{\boldsymbol{k}} e^{-\frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$\propto {z_{i,k}}^{(4\rho_i-1)} e^{-\sum_{\boldsymbol{k}} \frac{\rho_i}{c_{2i,\boldsymbol{k}}} \sum_{\boldsymbol{k}' \in \mathcal{V}_v(\boldsymbol{k})} z_{i,k'}}$$

$$\propto {z_{i,k}}^{(4\rho_i-1)} e^{-\rho_i z_{i,k} \sum_{\boldsymbol{k}'' \in \mathcal{V}_z(\boldsymbol{k})} \frac{1}{c_{2i,\boldsymbol{k}''}}}$$

$$\sim \mathcal{G}(4\rho_i, \rho_i \textstyle\sum_{\boldsymbol{k}'' \in \mathcal{V}_z(\boldsymbol{k})} c_{2\,i,\boldsymbol{k}''}^{-1})$$

Bayesian model with independent $\mathcal{IG}$ priors ($\sim$ Part 2)

- Posterior

independent/univariate model

$$p(\mathbf{C_2}, \mathbf{Z}, \mathbf{M} | \mathbf{Y}, \alpha, \beta) \propto \underbrace{p(\mathbf{Y}, \mathbf{M} | \mathbf{C_2})}_{\text{augmented likelihood}} \times \prod_{i=0}^{1} \underbrace{\prod_{\boldsymbol{k}} p(c_{2\,i,\boldsymbol{k}} | \alpha_i, \beta_i)}_{\text{independent } \mathcal{IG} \text{ priors}}$$

- Bayesian estimator $\rightarrow$ marginal posterior mean

$$\mathbf{C_2}^{\mathrm{MMSE}} = \mathbb{E}[\mathbf{C_2} | \mathbf{Y}, \rho] \quad \rho = (\rho_0, \rho_1)$$

$$\hat{\mathbf{C_2}}^{\mathrm{MMSE}} \approx \frac{1}{N_{mc} - N_{bi}} \sum_{s=N_{bi}+1}^{N_{mc}} \mathbf{C_2}^{(s)}$$

with $\{\mathbf{C_2}^{(s)}, \mathbf{M}^{(s)} \quad\quad \}_{s=1}^{N_{mc}}$ generated via an MCMC algorithm

Bayesian model with GaMRF prior

- Posterior

multivariate model

$$p(\mathbf{C_2}, \mathbf{Z}, \mathbf{M} | \mathbf{Y}, \rho) \propto \underbrace{p(\mathbf{Y}, \mathbf{M} | \mathbf{C_2})}_{\text{augmented likelihood}} \times \prod_{i=0}^{1} \underbrace{p(\mathbf{C_2}_i, \boldsymbol{Z}_i | \rho_i)}_{\text{GaMRF priors}}$$

- Bayesian estimator $\rightarrow$ marginal posterior mean

$$\mathbf{C_2}^{\mathrm{MMSE}} = \mathbb{E}[\mathbf{C_2} | \mathbf{Y}, \rho] \quad \rho = (\rho_0, \rho_1)$$

$$\hat{\mathbf{C_2}}^{\mathrm{MMSE}} \approx \frac{1}{N_{mc} - N_{bi}} \sum_{s=N_{bi}+1}^{N_{mc}} \mathbf{C_2}^{(s)}$$

with $\{\mathbf{C_2}^{(s)}, \mathbf{M}^{(s)}, \mathbf{Z}^{(s)}\}_{s=1}^{N_{mc}}$ generated via an MCMC algorithm

- Hyperparameters

$\rightarrow$ ρ not estimated here, manually fixed

Gibbs sampler with independent $\mathcal{IG}$ priors ($\sim$ Part 2)

1. Sampling of hidden means $\boldsymbol{\mu_k}$

   $p(\boldsymbol{\mu_k}|\mathbf{C_2}, \quad \mathbf{Y}, \rho)$ closed-form Gaussian distributions

2. Sampling of parameters $c_{2i,\boldsymbol{k}}$

   $p(c_{2i,\boldsymbol{k}}| \quad \mathbf{M}, \mathbf{Y}, \rho)$ closed-form inverse-gamma distributions

Gibbs sampler with GaMRF prior

1. Sampling of hidden means $\boldsymbol{\mu_k}$

   $p(\boldsymbol{\mu_k}|\mathbf{C_2}, \mathbf{Z}, \mathbf{Y}, \rho)$ closed-form Gaussian distributions

2. Sampling of parameters $c_{2i,\boldsymbol{k}}$

   $p(c_{2i,\boldsymbol{k}}|\mathbf{Z}, \mathbf{M}, \mathbf{Y}, \rho)$ closed-form inverse-gamma distributions

3. Sampling of auxiliary variables $\mathbf{Z}$

   $\mathrm{p}(z_{i,\boldsymbol{k}}|\mathbf{C_2}, \mathbf{M}, \mathbf{Y}, \rho)$ closed-form gamma distributions

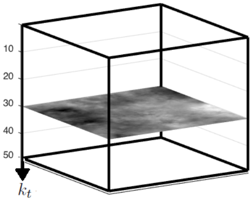
$\surd$ direct sampling for all conditional distributions

$\rightarrow$ efficient sampling scheme, tailored for large datasets

Numerical illustrations

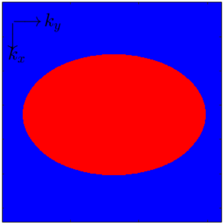
Synthetic multivariate multifractal image

→ sequence of heterogeneous 2D multifractal multiplicative cascades

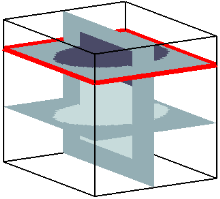


Sequence of MRWs

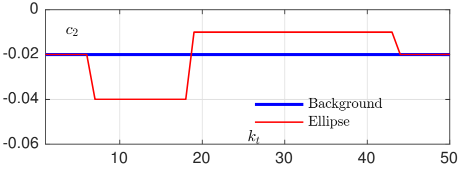
Frame ($t = 30$)



Piecewise const. $\boldsymbol{c_2}$



Piecewise const. $\boldsymbol{c_2}$



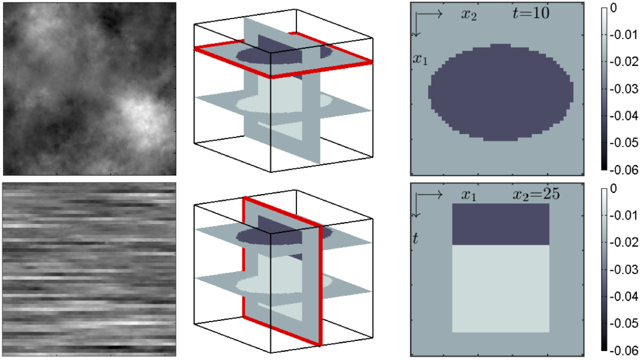
Temporal evolution of $\mathbf{C_2}_1$

- 3200 × 3200 × 50 pixel cube
- 50 × 50 × 50 patches of size 64 × 64
- analysis scales $j_1 = 1$, $j_2 = 2$
- parameters ρ_i by cross-validation

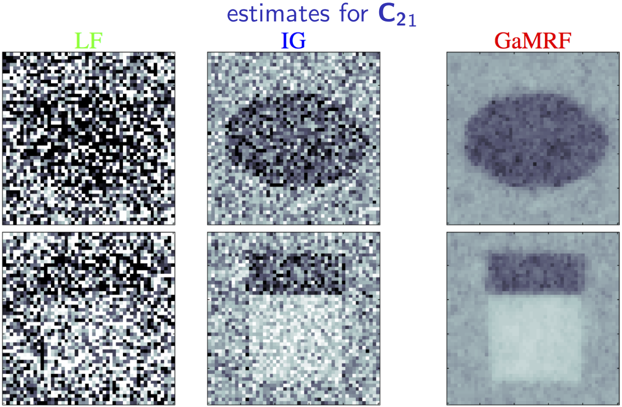
Estimation setting

Multifractality parameter: single realization

shown slices



Multifractality parameter: single realization

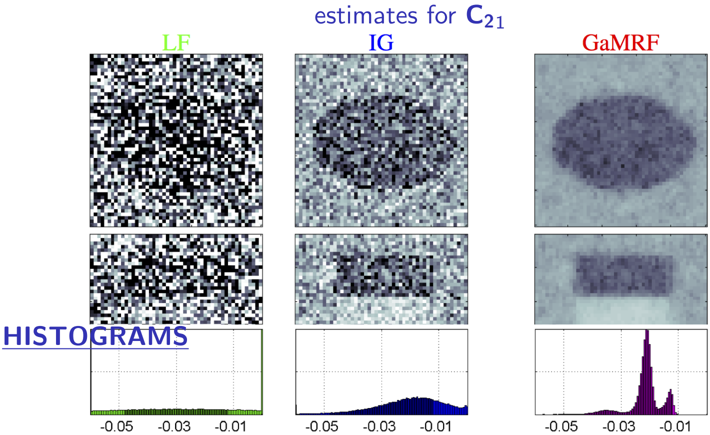


LF $\rightarrow$ strong spatial & temporal variability with no possible identification

IG $\rightarrow$ spatial and temporal variability but rough identification

GaMRF $\rightarrow$ spatial and/or temporal coherence with clear identification

Multifractality parameter: single realization



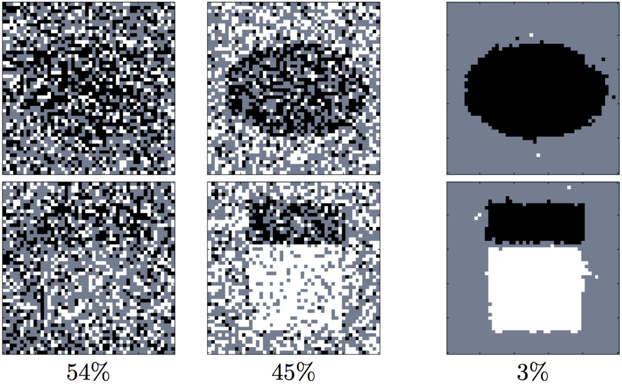
LF $\rightarrow$ strong spatial & temporal variability with no possible identification

IG $\rightarrow$ spatial and temporal variability but rough identification

GaMRF $\rightarrow$ spatial and/or temporal coherence with clear identification

Multifractality parameter: single realization

k-means classification

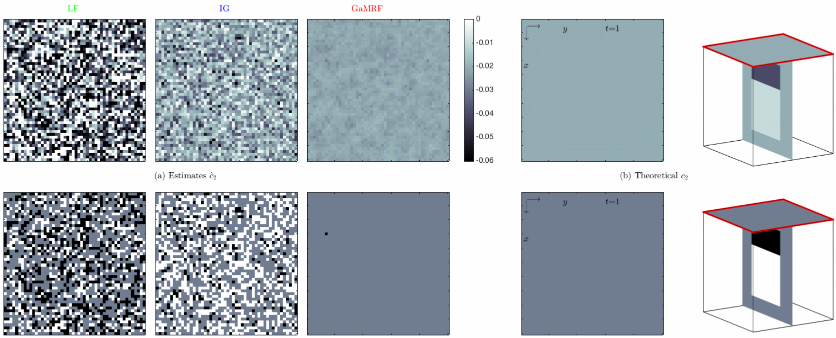


LF → strong spatial & temporal variability with no possible identification 54%

IG → spatial and temporal variability but rough identification 45%

GaMRF → spatial and/or temporal coherence with clear identification 3%

Multifractality parameter: single realization



(a) Estimates $\hat{c}_2$

(b) Theoretical c_2

(c) K-means classification

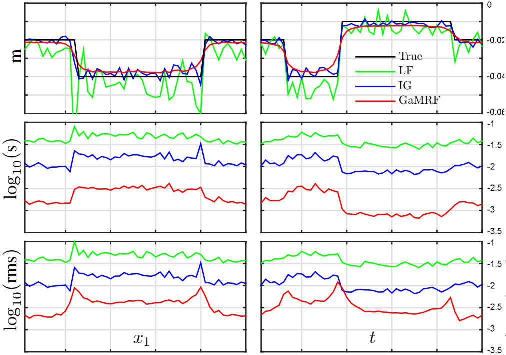
(d) Classes

LF $\rightarrow$ strong spatial & temporal variability with no possible identification 54%

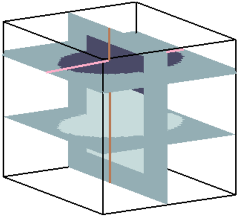
IG $\rightarrow$ spatial and temporal variability but rough identification 45%

GaMRF $\rightarrow$ spatial and/or temporal coherence with clear identification 3%

Multifractality parameter: estimation performance



(a) Line $(t = 10, x_2 = 25)$ (b) Line $(x_1 = 25, x_2 = 25)$



GaMRF:

→ smoothing of sharp transitions

→ reduction of STD/RMSE

	LF	IG	GaMRF
$\lvert b_{c_{21}} \rvert$	0.0057	**0.0017**	0.0023
$s_{c_{21}}$	0.038	0.011	**0.0016**
$r_{c_{21}}$	0.039	0.011	**0.0029**

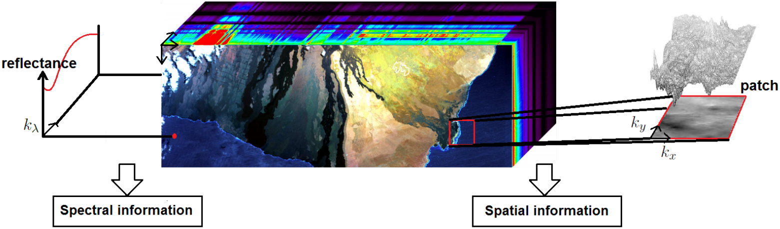
STD / RMSE:

– IG ∼ 4 times below LF

– GaMRF ∼ 10 times below LF

Multifractal analysis and hyperspectral imaging

- Hyperspectral (HS) imaging
  - observation of a scene at many different spectral bands ~ 100



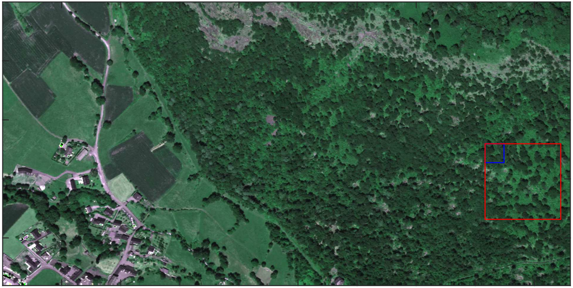
  - growing interest for spatial information
    (ever increasing spatial resolution of sensors ...)
- Multifractal analysis for textural information?
  - **X** limited attempts for HS $\sim$ fractal dimensions / H [Dong'08,Sun'06,Yin'12]

$\longrightarrow$ Multifractal features for textural information

$\longrightarrow$ Illustration for multivariate Bayesian estimation for multifractality

Analyzed hyperspectral image

- Madonna hyperspectral dataset [Sheeren'11]
  - dataset acquired over Villelongue by Hyspex hyperspectral scanner
  - 960×1952 pixels with spatial resolution of 0.5m
  - 160 spectral bands ranging from visible to near infrared

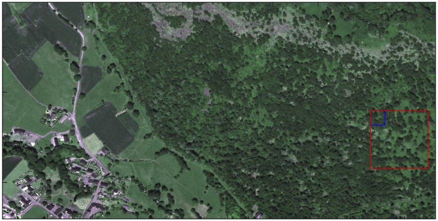


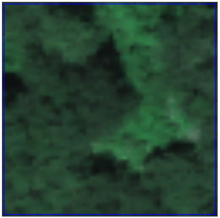
Does the model fit real-world data?

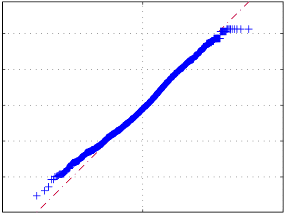
[Sheeren'11]

Does the model fit real-world data?

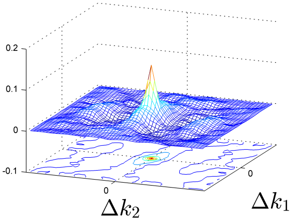
[Combrexelle-WHISPERS'15], [Sheeren'11]

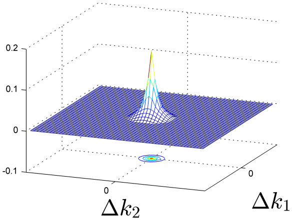


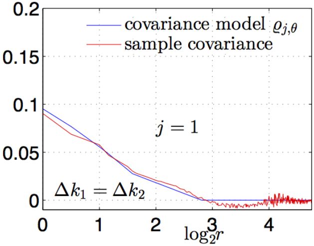




sample covariance and model (64×64):

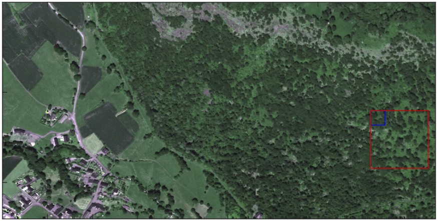


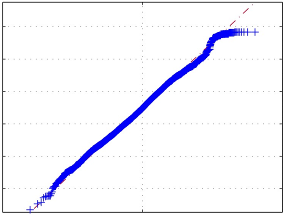




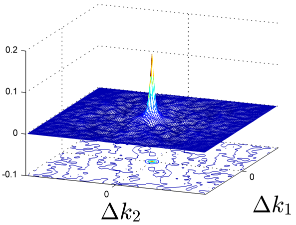
Does the model fit real-world data?

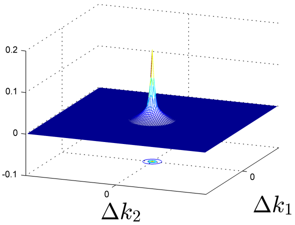
[Combrexelle-WHISPERS'15], [Sheeren'11]

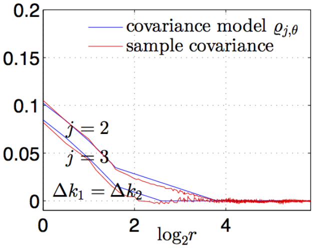




sample covariance and model (256×256):

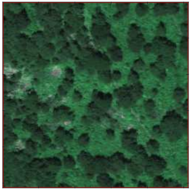




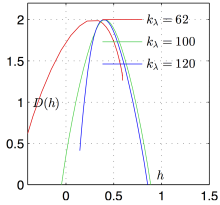


Spectral evolution of multifractal features

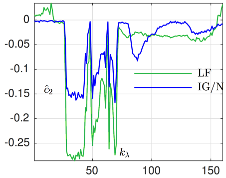
- Multifractal features vs. reflectance [Combrexelle-WHISPERS'15]



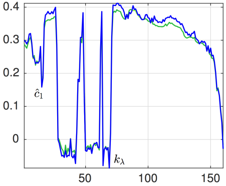
Forested area of interest



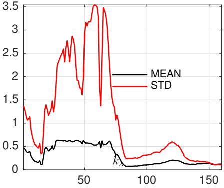
Multifractal spectrum



Evolution of $c_{21,\boldsymbol{k}}$



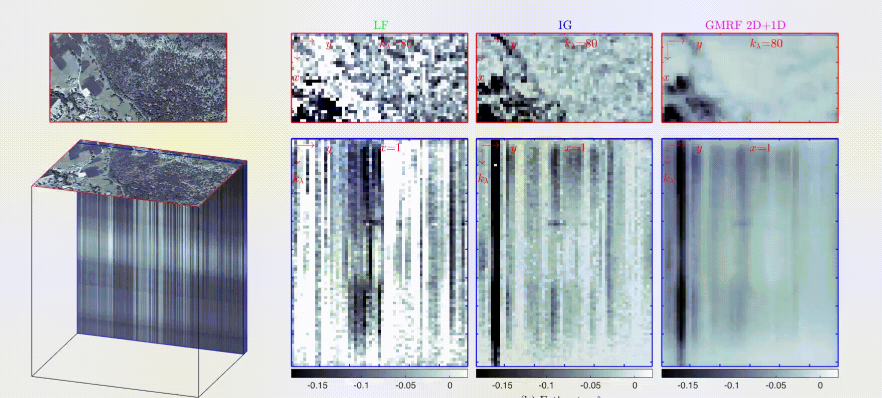
Evolution of $c_{11,\boldsymbol{k}}$



Average reflectance

Spatio-spectral evolution of multifractal features: $\mathbf{C}_{21}$

- Spatio-spectral evolution of $\mathbf{C}_{21}$

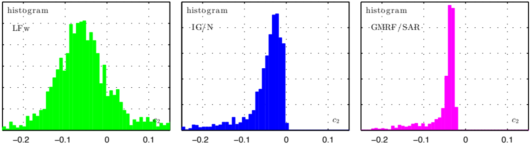


(a) Area of interest (framed)

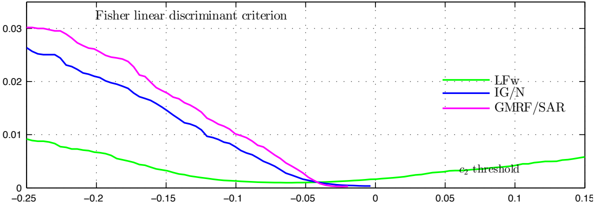
(b) Estimates $\hat{c}_2$

Spatio-spectral multifractal features: $\mathbf{C}_{21}$

- Histograms of $\mathbf{C}_{21}$ estimates ($k_\lambda = 114$)



- Fisher linear discriminant criteria of $\mathbf{C}_{21}$ estimates ($k_\lambda = 114$)



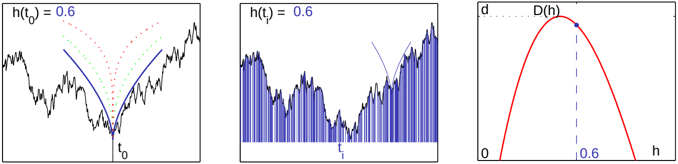
$\rightarrow$ GMRF $>$ IG LF

Macroscopic Brain

Conclusions

Multifractal analysis

– tool for global, geometric description of local regularity fluctuations



– assessment of statistics / dependence beyond 2nd order

– estimation via multifractal formalism using wavelet leaders

signal & image processing tool used in large panel of applications

e.g., physics, financial markets, geology, biology and biomedical (gene expression, fMRI, . . .), Art investigation, network traffic, . . .

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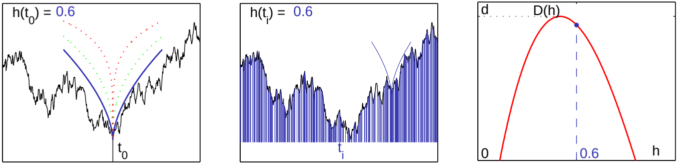

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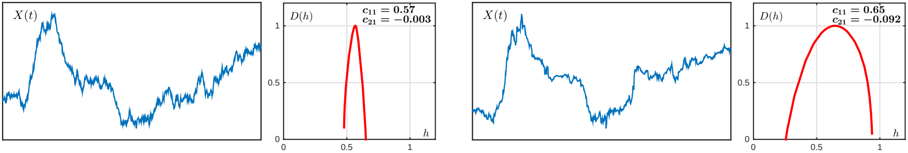

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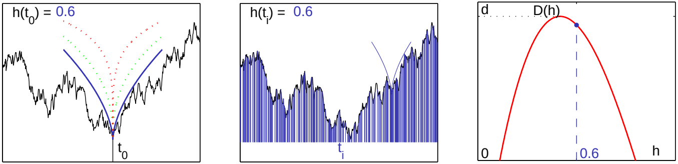

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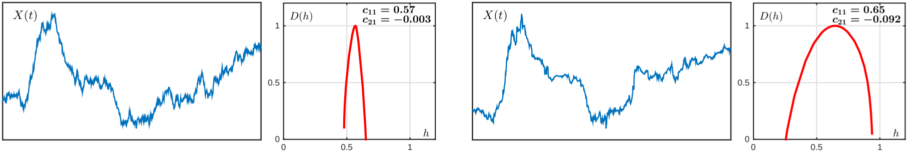

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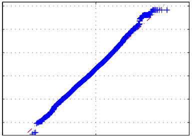
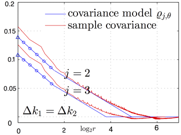
Conclusions

Multifractal analysis: Bayesian estimation

– Gaussian random field model for log-leaders





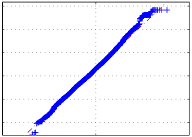
– Whittle approximation & data augmentation

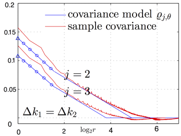
$\rightarrow$ data augmented Fourier domain likelihood $(\sim \mathcal{CN})$

Conclusions

Multifractal analysis: Bayesian estimation

– Gaussian random field model for log-leaders

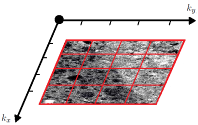


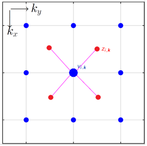


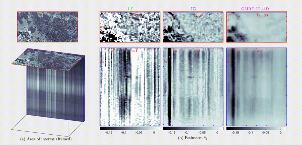
– Whittle approximation & data augmentation
$\rightarrow$ data augmented Fourier domain likelihood $(\sim \mathcal{CN})$

Multivariate model

– GaMRF joint prior for c_2 of different data components







$\rightarrow$ efficient inference via a Gibbs sampler (large data sets)
$\rightarrow$ significantly improved estimation performance (gain: factor ~ 10)

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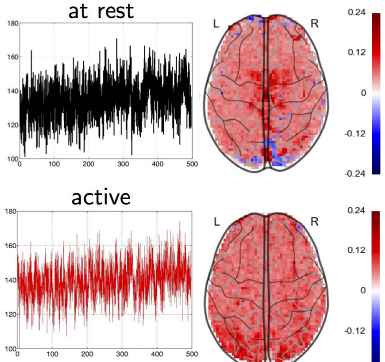
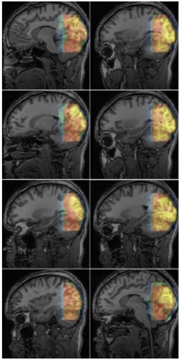
Scale-free dynamics and infraslow macroscopic brain activity

model	$1/f^{\beta}$	$\subset$	self-similar	$\subset$	multifractal
analysis	Fourier	$\longrightarrow$	wavelets	$\longrightarrow$	wavelet leaders
parameters	β		Hurst H		multifractal spectrum $D(h)$

[Shimizu'04,He'11,Ciuciu'12]



Ph. Ciuciu (CEA)



Ph. Ciuciu (CEA)

[Ciuciu-EMBC'17]

[Wendt-ISBI'18]

Collab. P. Ciuciu (CEA, NeuroSpin, France), P. Abry (ENS Lyon, France)

fMRI data: Experimental design and acquisition

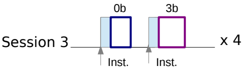
Verbal *n*-back working memory task ($n = 3$).

– serially presented upper-case letters (displayed 1s, separation 2s)

$\rightarrow$ Is letter same as that presented 3 stimuli before?

0-Back	1-Back	2-Back	3-Back
A C **X** F	C **D** **D** R	C **D** E **D** H	T **D** E K **D** Z

– each run: alternating sequence of 8 blocks



Data acquisition.

– resting-state fMRI images first: participant at rest, with eyes closed

– 543 scans (9min10s) / 512 scans (8min39s) for rest / task

– fMRI data acquisition at 3 Tesla (Siemens Trio, Germany)

– multi-band GE-EPI (TE=30ms, TR=1s, FA=61, MB=2) sequence (CMRR, USA), 3mm isotropic resolution, FOV of $192\times192\times144\text{mm}^3$

Shown results: $(-c_2)$ maps.

– for single subject (arbitrarily chosen from 40 participants).

Back

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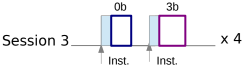

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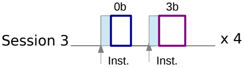

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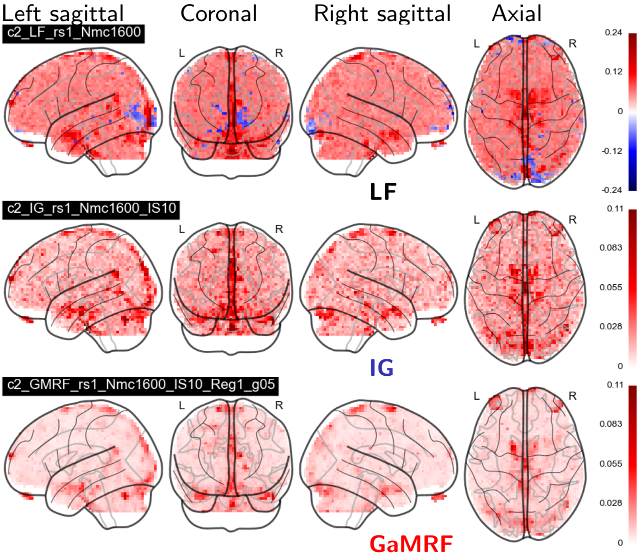
Resting-state analysis

$((-c_2)$ maps$)$



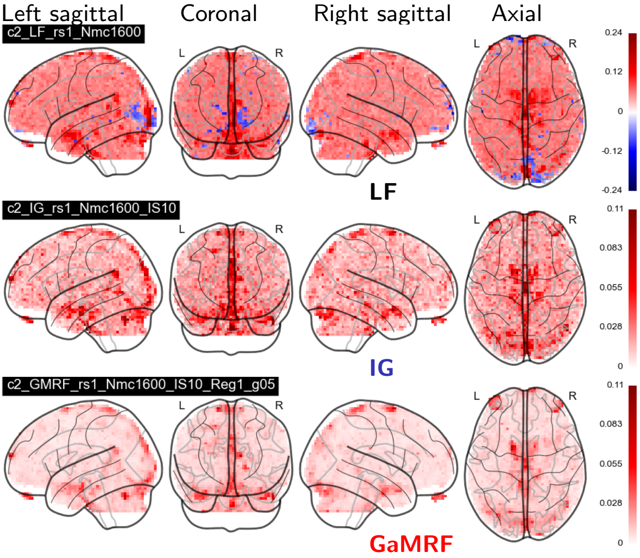
LF:
- poor estimation (var!)

IG & GaMRF:
- estimation var decrease

GaMRF:
- enhanced MF contrast

$\longrightarrow$ **significant MF in default mode network (DMN)**

Resting-state analysis ($(-c_2)$ maps)



LF:
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- increase of MF in DMN

GaMRF:
- enhanced MF contrast

scale-free dynamics in DMN for resting-state fMRI reported before, but for H only [He JNS'11].

$\longrightarrow$ *evidence for richer, MF resting state brain dynamics*

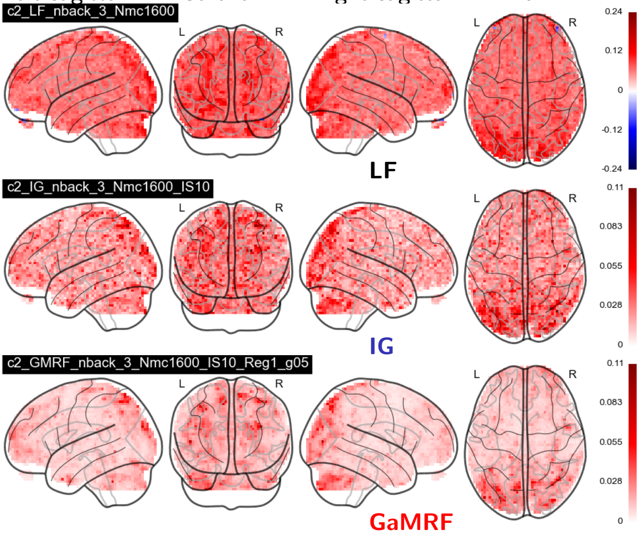
Back

$\longrightarrow$ **significant MF in default mode network (DMN)**

Task analysis: 3-back run

(($-c_2$) maps)

Left sagittal   Coronal   Right sagittal   Axial



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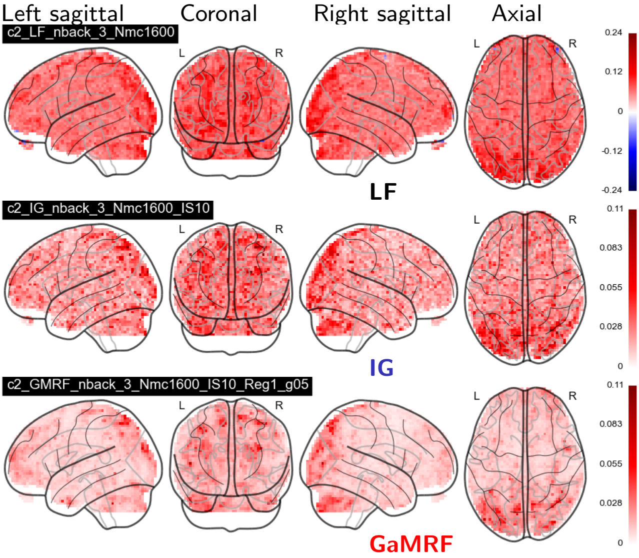
Back

$\longrightarrow$ **overall MF increase; working memory network (WMN), visual, sensory.**

Task analysis: 3-back run

$((-c_2)$ maps)



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overall increase in MF during task

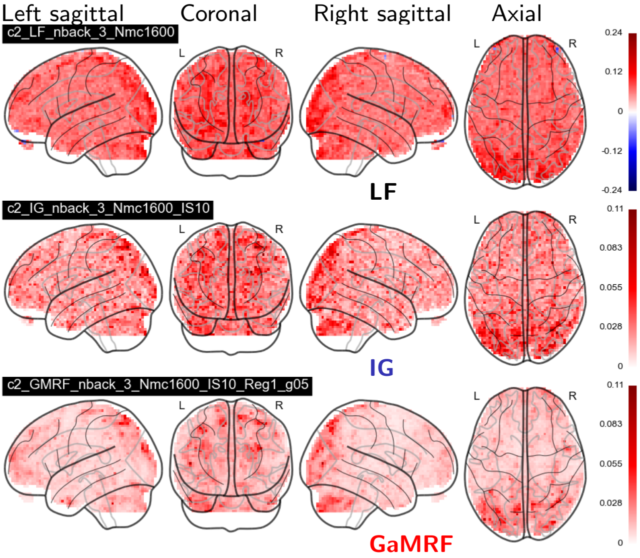
[Ciuciu FPhys'12]

Back

$\longrightarrow$ overall MF increase; working memory network (WMN), visual, sensory.

Task analysis: 3-back run

(($-c_2$) maps)



LF:
- poor estimation (var!)

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overall increase in MF during task

[Ciuciu FPhys'12]

GaMRF:
significant MF in
- bilateral parietal regions belonging to WMN
- occipital cortex (visual)
- cerebellum (sensory)

involved in task

Back

$\longrightarrow$ **overall MF increase; working memory network (WMN), visual, sensory.**

Log-cumulants

[Castaing'93]

Back

- For certain classes of processes:
  - $\mathbf{E}\ell(j,\cdot)^q \;=\; \mathbf{E}e^{q\ln\ell(j,\cdot)} \;=\; b_q 2^{j\zeta(q)}$

Log-cumulants

[Castaing'93]

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  - $\mathbf{E}\ell(j,\cdot)^q \;=\; \mathbf{E}e^{q\ln\ell(j,\cdot)} \;=\; b_q 2^{j^{\zeta(q)}}$

$\Rightarrow$ 2nd characteristic function of $\ln\ell(j,\cdot)$:

$C_p(j) = \mathrm{Cum}_p[\ln\ell(j,\cdot)]$: cumulant of order $p \geq 1$

$$\ln\mathbf{E}e^{q\ln\ell(j,\cdot)} = \sum_p C_p(j)\frac{q^p}{p!} = \ln b_q + \zeta(q)\ln 2^j$$

Log-cumulants

[Castaing'93]

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$$\ln \mathbf{E}e^{q\ln\ell(j,\cdot)} = \sum_p C_p(j)\frac{q^p}{p!} = \ln b_q + \zeta(q)\ln 2^j$$

$$= \sum_{p=1}^{\infty} c_{p0}\frac{q^p}{p!} + \sum_{p=1}^{\infty} c_{p1}\frac{q^p}{p!}\ln 2^j$$

$\Rightarrow\ \forall p \geq 1:$

$$\boxed{C_p(j) = c_{p_0} + c_{p_1}\ln 2^j}$$

Log-cumulants

[Castaing'93]

Back

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$\Rightarrow$ $\forall p \geq 1$: $\boxed{C_p(j) = c_{p_0} + c_{p_1}\ln 2^j}$

$\Rightarrow$ polynomial expansion

$$\boxed{\zeta(q) = \sum_{p=1}^{\infty} c_{p1}\frac{q^p}{p!}}$$

Summary: log-wavelet leaders statistical model

Linear regression

$\widehat{\text{Var}}[\boldsymbol{x}] \sim \mathcal{N}(\theta_2 + j\theta_1, \sigma^2) \quad x \sim \ln L_X(j,k) \quad \boldsymbol{\theta} \sim \boldsymbol{c_2}$

– independence assumption. marginals?
✓ single homogeneous large size data
X small size data piece

"Natural" model

$p(\boldsymbol{x}|\boldsymbol{\theta}) = \mathcal{N}_{\boldsymbol{x}}(0, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})$ $\quad \rho_{\boldsymbol{\theta}}(\tau) \mapsto \boldsymbol{\Sigma}_{\boldsymbol{\theta}}$

$\rho_{\boldsymbol{\theta}}(\tau) = \theta_1\rho_1(\tau) + \theta_2\rho_2(\tau)$ covariance model

– "direct" connection with cascades
X computation: $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}$? admissible c_2?
X no conjugate priors for θ_1, θ_2

$\downarrow$

Whittle approximation

$p(\boldsymbol{x}|\boldsymbol{\theta}) \approx \prod_m f_{\boldsymbol{\theta}}(\omega_m)^{-1} exp(-y_m^* y_m / f_{\boldsymbol{\theta}}(\omega_m))$

$\boldsymbol{y} = DFT(\boldsymbol{x})$

$f_{\boldsymbol{\theta}}(\omega) = \theta_1 f_1(\omega) + \theta_2 f_2(\omega)$ PSD model

✓ computation
Gibbs V.1
X no conjugate priors for θ_1, θ_2
$\rightarrow$ Metropolis-Hasting moves

$\downarrow$

Fourier domain model

$p(\boldsymbol{y}|\boldsymbol{\theta}) = \mathcal{CN}_{\boldsymbol{y}}(0, \text{diag}(f_{\boldsymbol{\theta}}(\omega_m)))$

$\rightarrow$ data $\boldsymbol{y} = DFT(\boldsymbol{x})$ replace data $\boldsymbol{x}$

– complex Gaussian & factorized in y_m
X Var $(y_m) \propto (\theta_1 + \theta_2)$
$\rightarrow$ no conjugate priors for θ_1, θ_2

$\downarrow$

Data augmented model using latent variables $\boldsymbol{\mu}$

$p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{\theta}) = \mathcal{CN}_{\boldsymbol{y}}(\boldsymbol{\mu}, \theta_1 \text{diag}(f_1(\omega_m)))$
$\times \mathcal{CN}_{\boldsymbol{\mu}}(0, \theta_2 \text{diag}(f_2(\omega_m)))$

$p(\boldsymbol{y}|\boldsymbol{\theta}) = \int p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{\theta}) d\boldsymbol{\mu}$

✓ θ_1 and θ_2 are separable
$\rightarrow$ conjugate $\mathcal{IG}$ priors for θ_1, θ_2
Gibbs V.2 $\rightarrow$ fast sampling from closed-form conditionals

Back

Model: time-domain statistical model of log-leaders

1. Marginal distribution of log-leaders approximated by Gaussian

$$\boxed{l(j,\cdot,\cdot) = \ln L(j,\cdot,\cdot) \;\sim\; \mathcal{N}\left(\cdot, c_2^0 + c_2 \ln 2^j\right)}$$

2. Intra-scale parametric covariance model

$$\boxed{\mathrm{Cov}[l(j,k), l(j,k+\Delta r)] \approx \varrho_j(\Delta r; \boldsymbol{v}), \quad \boldsymbol{v} = (c_2, c_2^0)}$$

- Likelihood of centered log-leaders $\boldsymbol{l}_j$ stacked in $\boldsymbol{l} = [\boldsymbol{l}_{j_1}^T, ..., \boldsymbol{l}_{j_2}^T]^T$

  $\rightarrow$ scale-wise product of Gaussian likelihoods

$$p(\boldsymbol{l}|\boldsymbol{v}) \propto \prod_{j=j_1}^{j_2} |\boldsymbol{\Sigma}_{j,\boldsymbol{v}}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\boldsymbol{l}_j^T \boldsymbol{\Sigma}_{j,\boldsymbol{v}}^{-1} \boldsymbol{l}_j\right), \text{ with } \boldsymbol{\Sigma}_{j,\boldsymbol{v}} \text{ induced by } \varrho_j(\Delta r; \boldsymbol{v})$$

  X evaluation of $p(\boldsymbol{l}|\boldsymbol{v})$ numerically instable

[TIP15]

Back

Model: Whittle approximation

- Evaluation of the Gaussian likelihood in the spectral domain

$$p_W(\boldsymbol{l}|\boldsymbol{v}) \propto \prod_{j=j_1}^{j_2} |\boldsymbol{\Gamma}_{j,\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}_j^H \boldsymbol{\Gamma}_{j,\boldsymbol{v}}^{-1} \boldsymbol{y}_j\right)$$

  - $\boldsymbol{y}_j$ Fourier coefficients of $\boldsymbol{l}_j$
  - $\boldsymbol{\Gamma}_{j,\boldsymbol{v}}$ parametric spectral density associated with $\varrho_j(\Delta r; \boldsymbol{v})$

    $\rightarrow$ closed-form expression via Hankel transform

$$\boldsymbol{\Gamma}_{j,\boldsymbol{v}} = c_2 \, \mathbf{F}_{1,j} + c_2^0 \, \mathbf{F}_{2,j}, \quad \mathbf{F}_{i,j} = \mathrm{diag}(\mathbf{f}_{i,j})$$

- Estimation of $\boldsymbol{v}$ embedded in a Bayesian framework
  - space-domain likelihood (approximated) + common priors

  X non-standard posterior distribution $\rightarrow$ acceptance/reject moves

[TIP15]

Back

Model: Fourier-domain statistical model

- Whittle approximation

$$p_W(\boldsymbol{l}|\boldsymbol{v}) \propto \prod_{j=j_1}^{j_2} |\boldsymbol{\Gamma}_{j,\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}_j^H \boldsymbol{\Gamma}_{j,\boldsymbol{v}}^{-1} \boldsymbol{y}_j\right)$$

  - $\boldsymbol{y}_j$ Fourier coefficients of $\boldsymbol{l}_j$
  - $\boldsymbol{\Gamma}_{j,\boldsymbol{v}} = c_2\, \mathbf{F}_{1,j} + c_2^0\, \mathbf{F}_{2,j}$ parametric spectral density

$$\Updownarrow$$

- Generative model for $\boldsymbol{y} = [\boldsymbol{y}_{j_1}^T, ..., \boldsymbol{y}_{j_2}^T]^T$

$$p(\boldsymbol{y}|\boldsymbol{v}) \propto |\boldsymbol{\Gamma}_{\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}^H \boldsymbol{\Gamma}_{\boldsymbol{v}}^{-1} \boldsymbol{y}\right)$$

  - complex Gaussian model $\boldsymbol{y} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Gamma}_{\boldsymbol{v}})$
  - $\boldsymbol{\Gamma}_{\boldsymbol{v}} = c_2 \boldsymbol{F}_1 + c_2^0 \boldsymbol{F}_2$ and $\boldsymbol{F}_i = \text{block}(\boldsymbol{F}_{i,j_1}, \ldots, \boldsymbol{F}_{i,j_2})$

X model non-separable in (c_2, c_2^0)

Back

Model: Reparametrization

- Non-separable constraints on (c_2, c_2^0)

$$\boldsymbol{v} \in \mathcal{A} = \{(c_2, c_2^0) \in \mathbb{R}_\star^- \times \mathbb{R}_\star^+ | \boldsymbol{\Gamma}_{\boldsymbol{v}} = c_2 \mathbf{F}_1 + c_2^0 \mathbf{F}_2 \text{ positive-definite}\}$$

- Design of a linear diffeomorphism ψ

  1 mapping joint constraints into independent positivity constraints

$$\begin{aligned} \psi \colon & \mathcal{A} \to \mathbb{R}_\star^{+2} \\ \colon & \boldsymbol{v} \mapsto \psi(\boldsymbol{v}) \triangleq \boldsymbol{v} \end{aligned}$$

  2 yielding more convenient likelihood

$$p(\boldsymbol{y}|\boldsymbol{v}) \propto |\boldsymbol{\Gamma}_{\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}^H \boldsymbol{\Gamma}_{\boldsymbol{v}}^{-1} \boldsymbol{y}\right) \quad \text{with}$$

$$\text{for } \boldsymbol{v} \in \mathbb{R}_\star^{+2} \begin{cases} \boldsymbol{\Gamma}_{\boldsymbol{v}} = \tilde{\theta}_1 \tilde{\boldsymbol{F}}_1 + \tilde{\theta}_2 \tilde{\boldsymbol{F}}_2 & \text{positive-definite} \\ \\ \tilde{\theta}_i \tilde{\boldsymbol{F}}_i & \text{positive-definite} \end{cases}$$

$\rightarrow$ separability of the likelihood via data augmentation

Back

Model: Data augmentation

- Definition of an augmented model

$$\begin{cases} \boldsymbol{y}|\boldsymbol{\mu},\tilde{\theta}_2 \sim \mathcal{CN}(\boldsymbol{\mu},\tilde{\theta}_2\tilde{\boldsymbol{F}}_2) & \text{observed data} \\ \boldsymbol{\mu}|\tilde{\theta}_1 \sim \mathcal{CN}(\boldsymbol{0},\tilde{\theta}_1\tilde{\boldsymbol{F}}_1) & \text{hidden mean} \end{cases}$$

with

$$p(\boldsymbol{y}|\boldsymbol{v}) = \int p(\boldsymbol{y},\boldsymbol{\mu}|\boldsymbol{v})d\boldsymbol{\mu}$$

- Virtues of the augmented likelihood $p(\boldsymbol{y},\boldsymbol{\mu}|\boldsymbol{v})$

$$p(\boldsymbol{y},\boldsymbol{\mu}|\boldsymbol{v}) \propto \tilde{\theta}_2^{\,-N_Y} \exp\left(-\frac{1}{\tilde{\theta}_2}(\boldsymbol{y}-\boldsymbol{\mu})^H\tilde{\boldsymbol{F}}_2^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right) \times \tilde{\theta}_1^{\,-N_Y} \exp\left(-\frac{1}{\tilde{\theta}_1}\boldsymbol{\mu}^H\tilde{\boldsymbol{F}}_1^{-1}\boldsymbol{\mu}\right)$$

$\surd$ separable in $(\tilde{\theta}_1,\tilde{\theta}_2)$

$\surd$ conjugate to inverse-gamma priors

Back

MCMC algorithm

- Strategy of Gibbs sampler
  - iterative sampling according to conditional laws
  - non-standard conditional laws $\rightarrow$ Metropolis-within-Gibbs
  - computation of acceptance ratio at each iteration

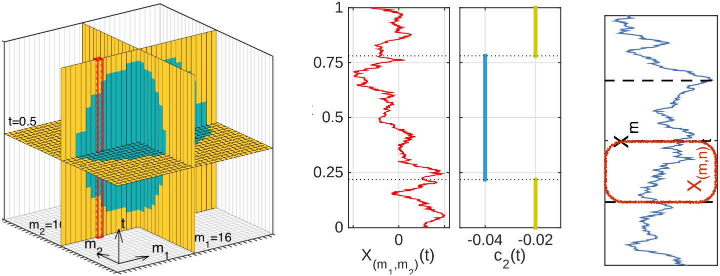
$$r_{c_2} = \sqrt{\frac{\det \boldsymbol{\Sigma}(\boldsymbol{v}^{(t)})}{\det \boldsymbol{\Sigma}(\boldsymbol{v}^{(\star)})}} \times \prod_{j=j_1}^{j_2} \exp\left(-\frac{1}{2}\boldsymbol{l}_j^T \left(\boldsymbol{\Sigma}_{j,\boldsymbol{v}}(\boldsymbol{v}^{(\star)})^{-1} - \boldsymbol{\Sigma}_{j,\boldsymbol{v}}(\boldsymbol{v}^{(t)})^{-1}\right)\boldsymbol{l}_j\right)$$

Time block wise estimation (2D+time)

- Synthetic multifractal time series: Multifractal Random Walk

  $\sim$ Mandelbrot's celebrated multiplicative cascades

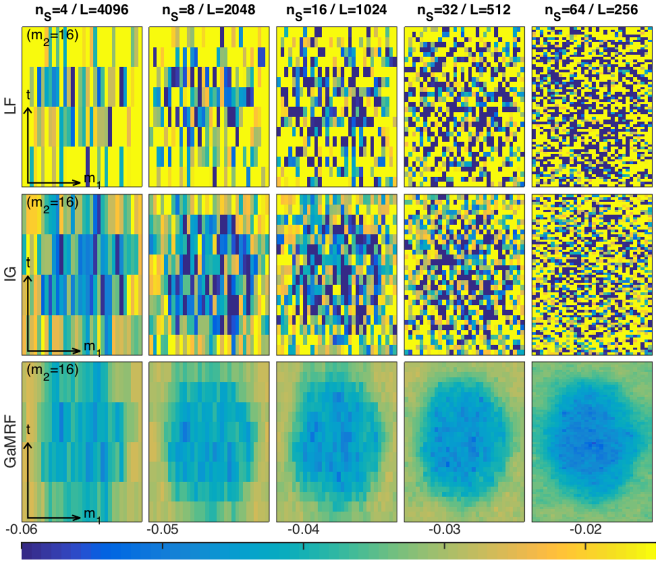
- collection of 32×32 time series of length $N = 2^{14}$
  - piece-wise constant $c_2 \in \{-0.02, -0.04\}$ along time



- Comparison of estimators for c_2
  - $n_S = 2^{2,\dots,6}$ windows of lengths $L = \{2^{12}, 2^{11}, 2^{10}, 2^9, 2^8\}$
  - LF – univariate linear regression based estimation
  - IG – univariate Bayesian estimation [TIP15,ICASSP16]
  - GaMRF – joint Bayesian estimator

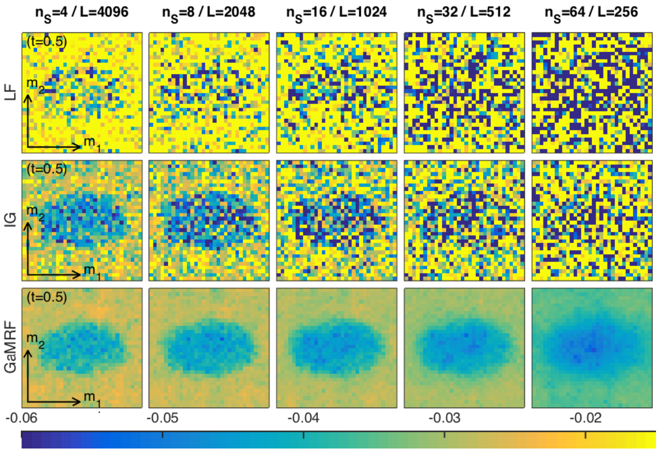
Time block wise estimation (2D+time)

estimates for c_2: temporal evolution at slice $m_2 = 16$



Time block wise estimation (2D+time)

estimates for c_2: spatial cross-section at $t = 0.5$



Time block wise estimation (2D+time)

RMSE (50 independent realizations)

n_S / L	4 / 2^{12}	8 / 2^{11}	16 / 2^{10}	32 / 2^{9}	64 / 2^{8}
LF	0.020	0.026	0.037	0.058	0.102
IG	0.011	0.013	0.018	0.024	0.036
GaMRF	**0.008**	**0.008**	**0.009**	**0.009**	**0.013**